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UNIFORM ANALYTICITY OF ORTHOGONAL PROJECTIONS

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ABSTRACT. Let X denote the circle T or the interval $[-1,1]$, and let $d\mu$ denote a nonnegative, absolutely continuous measure on X. Under what conditions does the Gram-Schmidt procedure in the weighted space $L^2(X, \omega^2\, d\mu)$ depend analytically on the logarithm of the weight function ω? In this paper, we show that, in numerous examples of interest, $\log\omega \in BMO$ is a sufficient (often necessary!) condition for analyticity of the Gram-Schmidt procedure. These results are then applied to establish the local analyticity of certain infinite-dimensional Toda flows.

1. INTRODUCTION

Let X denote the circle $\mathbf{T}$ or the interval $[-1,1]$, let $d\mu$ be a nonnegative measure on X which is absolutely continuous with respect to Lebesgue measure, and let $L(0)$ denote the complex Hilbert space $L^2(X, d\mu)$. Let ω be a nonnegative $d\mu$-measurable function on X such that $\omega^2 + \omega^{-2} \in L^1(X, d\mu)$ and let $\beta = \log\omega$; clearly also $\beta \in L^1(X, d\mu)$. Let $L(\beta)$ denote the complex weighted Hilbert space $L^2(X, \omega^2\, d\mu)$ and, for each nonnegative integer n, let $H_n(\beta)$ denote the closure of the polynomials (in the case $X = \mathbf{T}$, *trigonometric* polynomials) of degree at most n in $L(\beta)$. Let $S_n(\beta)$ denote the orthogonal projection of $L(\beta)$ onto $H_n(\beta)$. We wish to study the dependence of the family of operators $\langle S_n(\beta): n \in \mathbf{N}\rangle$ upon the functional parameter β. Each $S_n(\beta)$ is a bounded operator on $L(\beta)$, which varies with β, so to facilitate our study, we "lift" each operator $S_n(\beta)$ to $L(0)$ by means of the operator M_ω of pointwise multiplication by ω, which is an isometry from $L(\beta)$ to $L(0)$. If we define, for each nonnegative integer n, the operator $Q_n(\beta) = M_\omega S_n(\beta) M_\omega^{-1}$, then we see that the $L(0)$-boundedness of $Q_n(\beta)$ is equivalent to the $L(\beta)$-boundedness of $S_n(\beta)$, and the operator norms are equal. In fact, $Q_n(\beta)$ is easily seen to be the self-adjoint projection of $L(0)$ onto $M_\omega H_n(\beta) \subseteq L(0)$.

We would like to determine conditions on β under which the family of operators $\langle Q_n(\beta)\rangle$ depends analytically (in a sense to be made precise) upon the functional parameter β. In the specific examples which we consider, it is

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difficult if not impossible to write down the operators $S_n(\beta)$ and $Q_n(\beta)$ explicitly. On the other hand, the "base projection" $S_n(0)$ is an integral operator whose kernel is comparatively easy to write down. Rather than study the operators $\langle Q_n(\beta)\rangle$ directly, it is much more convenient to work with the family of operators $\langle P_n(\beta)\rangle$ defined by

$$P_n(\beta) = M_\omega S_n(0) M_\omega^{-1}. \tag{1.1}$$

For each nonnegative integer n, $P_n(\beta)$ is an oblique (i.e., non-self-adjoint) projection from $L(0)$ onto $M_\omega H_n(\beta) \subseteq L(0)$, and its adjoint, $P_n(\beta)^*$, is simply $P_n(-\beta)$.

A remarkable formula due to Kerzman and Stein ([8]) shows that, in fact, $\langle Q_n(\beta)\rangle$ depends analytically on β whenever $\langle P_n(\beta)\rangle$ depends analytically on β. Moreover, the analytic dependence of $\langle P_n(\beta)\rangle$ upon β is essentially equivalent to a uniform weighted norm inequality of the form

$$\int_X |S_n(0)f(x)|^2 \omega^2(x)\,d\mu(x) \le C \int_X |f(x)|^2 \omega^2(x)\,d\mu(x) \tag{1.2}$$

where C is a constant independent of n and f.

In addition, we would like to determine the "space of uniform holomorphy" for the family $\langle Q_n(\beta)\rangle$, i.e. the largest Banach function space on which the family $\langle Q_n\rangle$ is analytic at the origin. In practice, this amounts to determining necessary and sufficient conditions on the function β such that

$$\int_X |[M_\beta, S_n(0)]f(x)|^2\,d\mu(x) \le C \int_X |f(x)|^2\,d\mu(x) \tag{1.3}$$

where M_β is the operator of pointwise multiplication by β, and C is a constant independent of n and f.

Our work in this paper has been inspired in part by the related work of Coifman and Rochberg in [2], and by questions arising from the study of Toda flows in infinite dimensions (see, for example, [4]).

In this paper, we consider a number of examples. In the case $X = \mathbf{T}$, $d\mu = d\theta$, the uniform weighted norm inequality (1.2), and the uniform commutator estimate (1.3), are equivalent to the same inequalities with $S_n(0)$ replaced by the conjugate function. In this simplest example, the space of uniform holomorphy for $\langle Q_n(\beta)\rangle$ is easily seen to be the space of functions of bounded mean oscillation on $\mathbf{T}$. In the case $X = [-1, 1]$, $d\mu =$ Lebesgue measure weighted by a Jacobi weight, the uniform weighted norm inequality (1.2) follows from a weighted norm inequality for the Hilbert transform. In this case, we prove that $\langle Q_n(\beta)\rangle$ depends analytically on β when β is in a neighborhood of 0 in the space $\mathbf{BMO}([-1, 1])$. As an application of this result, we show that the Toda flow corresponding to the measure $\omega(x)^{2t}d\mu(x)$ on $[-1, 1]$ is analytic in t in a neighborhood of the origin provided that $\beta \in \mathbf{BMO}([-1, 1])$.

We conjecture that $\mathbf{BMO}([-1, 1])$ is the space of uniform holomorphy for $\langle Q_n\rangle$ in the case where $d\mu =$ Lebesgue measure weighted by a Jacobi weight.

We consider the example $X = [0, \pi]$, $d\mu = d\theta$, $S_n = n$th partial sum operator for cosine series, and show that, in this example, **BMO**$([0, \pi])$ is the space of uniform holomorphy for $\langle Q_n \rangle$. From this result it is immediate that the conjecture is true for $d\mu(x) = (1 - x^2)^{-1/2} dx$. Classical equiconvergence results for Jacobi series and cosine series (see [10]) suggest that the conjecture is probably true in general.

2. Uniform analyticity of projections: General setting

Let $(X, d\mu)$ be a σ-finite measure space; set $L(0) = L^2(X, d\mu)$. Suppose that ω is a nonnegative real-valued function such that $\omega^2 + \omega^{-2} \in L^1_{\text{loc}}(X, d\mu)$. We write $\beta = \log \omega$ and observe that $\beta \in L^1_{\text{loc}}(X, d\mu)$. Let $\mathbf{N}$ denote the set of nonnegative integers, and suppose that for each $n \in \mathbf{N}$, $H_n(0)$ is a closed subspace of $L(0)$. We assume that

$L(0, \beta) = L^2(X, (\omega^2 + \omega^{-2}) d\mu)$ is dense in $L(0)$ and

$H_n(0, \beta) = L(0, \beta) \cap H_n(0)$ is dense in $H_n(0)$, for each $n \in \mathbf{N}$.

We define the spaces:

$L(\beta) = L^2(X, \omega^2 d\mu)$,

$\mathscr{L}(\beta) =$ bounded linear operators on $L(\beta)$,

$H_n(\beta) =$ closure of $H_n(0, \beta)$ in $L(\beta)$, for each $n \in \mathbf{N}$.

We assume that the foregoing are *complex* Hilbert spaces.

For each $n \in \mathbf{N}$, let $S_n(\beta) \in \mathscr{L}(\beta)$ be the selfadjoint projection of $L(\beta)$ onto $H_n(\beta)$. We wish to study the dependence of the operators $\langle S_n(\beta) : n \in \mathbf{N} \rangle$ upon the functional parameter β. To facilitate our study, we "lift" each operator $S_n(\beta)$ back to $\mathscr{L}(0)$ by means of the operator M_ω of pointwise multiplication by ω, to wit: for each $n \in \mathbf{N}$, define $Q_n(\beta) = M_\omega S_n(\beta) M_\omega^{-1}$. Then $Q_n(\beta)$ is the self-adjoint projection of $L(0)$ onto $M_\omega H_n(\beta) \subseteq L(0)$, and $\|Q_n(\beta)\|_{\mathscr{L}(0)} = \|S_n(\beta)\|_{\mathscr{L}(\beta)}$.

We would like to formulate a clear conception of the "analytic dependence" of the family of operators $\langle Q_n(\beta) : n \in \mathbf{N} \rangle$ upon the functional parameter β. To this end, we make the following definitions.

Definitions. Let B be a real Banach space, $\mathscr{L}(H)$ the space of bounded linear operators on a complex Hilbert space H. For $n \in \mathbf{N}$, let $T_n : B \to \mathscr{L}(H)$ be an operator-valued function on B.

(a) $\langle T_n : n \in \mathbf{N} \rangle$ is said to be *uniformly* (*real-*) *analytic* in a neighborhood of 0 in B if and only if there is a constant $C > 0$ such that, whenever $b \in B$ with $\|b\|_B \leq C$ and whenever $f \in H$, we have

$$T_n(b) f = \sum_{k=0}^{\infty} \Lambda_{n,k}(b, \dots, b, f), \quad \text{for all } n \in \mathbf{N}, \tag{2.1}$$

where $\Lambda_{n,k}$ is a bounded, $(k+1)$-multilinear operator from $B^k \times H \to H$ which satisfies an estimate of form

$$\|\Lambda_{n,k}(b, \dots, b, f)\|_H \leq C_0^{k+1} \|b\|_B^k \|f\|_H \tag{2.2}$$

where C_0 is independent of b, f, n, and k.

(b) Let $\mathbf{B}$ denote the complexification of B. $\langle T_n : n \in \mathbf{N}\rangle$ is said to be *uniformly holomorphic* in a neighborhood of 0 in $\mathbf{B}$ if and only if there is a neighborhood of 0 in $\mathbf{B}$ to which each T_n can be extended, and there is a constant $C > 0$ such that, whenever $b \in \mathbf{B}$ with $\|b\|_{\mathbf{B}} \leq C$ and whenever $f \in H$, we have (2.1) and (2.2) with '$\mathbf{B}$' in place of 'B'.

(c) $\mathbf{B}$ is called the *space of uniform holomorphy at* 0 for the family $\langle T_n : n \in \mathbf{N}\rangle$ if and only if $\langle T_n\rangle$ is uniformly holomorphic in a neighborhood of 0 in $\mathbf{B}$ *and* a necessary and sufficient condition for

$$\sup_n \|\Lambda_{n,1}(b, \cdot)\|_{\mathscr{L}(H)} \tag{2.3}$$

to be finite is that $b \in \mathbf{B}$.

We pause to observe that the operator $\Lambda_{n,k}(b, \ldots, b, \cdot)$ occurring in (2.1) is just the kth Gâteaux (or Frechét) differential of T_n at 0 in the direction b (see, for example, [1, Chapter 2]). We give one equivalent formulation of the notion of uniform holomorphy in terms of Gâteaux differentiability, which will be useful in practice.

Proposition 2.1. *Let $\mathbf{B}$ be a complex Banach space, $\mathscr{L}(H)$ the space of bounded linear operators on a complex Hilbert space H. For $n \in \mathbf{N}$, let $T_n : \mathbf{B} \to \mathscr{L}(H)$ be an operator-valued function on $\mathbf{B}$. Then $\langle T_n\rangle$ is uniformly holomorphic in a neighborhood of 0 in $\mathbf{B}$ if and only if there exists a neighborhood U of 0 in $\mathbf{B}$ on which each T_n is Gâteaux differentiable and there exists a constant C such that for all $n \in \mathbf{N}$ and for all $\beta \in U$, $\|T_n(\beta)\|_{\mathscr{L}(H)} \leq C$.*

Proof. The proof is an easy modification of the proof of Theorem 2.3.3 of [1] and will be omitted. □

We can now formulate our general problem precisely, as follows: we wish to identify the space of uniform holomorphy at 0 for the family $\langle Q_n : n \in \mathbf{N}\rangle$. In practice it is often difficult to characterize the projections $S_n(\beta)$ and $Q_n(\beta)$; it is much more convenient to work with the operators $\langle P_n(\beta) : n \in \mathbf{N}\rangle$ defined by

$$P_n(\beta) = M_\omega S_n(0) M_\omega^{-1}. \tag{2.4}$$

For $n \in \mathbf{N}$, $P_n(\beta)$ is an oblique projection from $L(0)$ onto $M_\omega H_n(\beta)$ with adjoint $P_n(\beta)^* = P_n(-\beta)$. The formula of Kerzman and Stein (see [8, §3.4]) makes it possible to deduce the uniform holomorphy of $\langle Q_n\rangle$ from that of $\langle P_n\rangle$, and greatly simplifies the computation of the Gâteaux differentials of the operators $\langle Q_n\rangle$:

Proposition 2.2 (Kerzman-Stein Formula). *Let H be a complex Hilbert space, K a closed subspace. Let Q be the self-adjoint projection of H onto K, and let P be a bounded oblique projection from H onto K. Then:*

(a) *$I + (P - P^*)$ is invertible.*

(b) $Q = P[I + (P - P^*)]^{-1}$;

(c) *whenever* c_0 *and* M *are positive constants with* $\|P - P^*\| \leq c_0$ *and* $M > \frac{1}{2}(c_0^2 - 1)$, *the series*

$$P\left\{\frac{1}{M+1}\sum_{k=0}^{\infty}\left[\frac{MI-(P-P^*)}{M+1}\right]^k\right\} \tag{2.5}$$

converges in the operator norm topology to Q.

Proof. The operator $P-P^*$ is skew-adjoint, so its spectrum is purely imaginary. In particular, -1 is not in the spectrum of $P-P^*$, from which (a) follows.

Clearly $QP = P$ and $PQ = Q$. Now let $h \in H$ and let $(\cdot|\cdot)$ denote the inner product on H. We have

$$\begin{aligned} (QP^*h|h) &= (P^*h|Qh) \quad \text{since } Q^* = Q \\ &= (h|PQh) \\ &= (h|Qh) \quad \text{since } PQ = Q \\ &= (Qh|h) \quad \text{since } Q^* = Q. \end{aligned} \tag{2.6}$$

Hence $QP^* = Q$; consequently

$$P = Q + (P-Q) = Q + (QP - QP^*) = Q[I + (P-P^*)] \tag{2.7}$$

whereupon we obtain (b).

It is tempting to expand $[I + (P-P^*)]^{-1}$ in a Neumann series, but we do not know that $\|P-P^*\| < 1$. Instead we proceed as follows. For any constant $M > 0$, we have

$$\begin{aligned} I + (P-P^*) &= I + MI - [MI - (P-P^*)] \\ &= (1+M)\left[I - \frac{MI-(P-P^*)}{1+M}\right]. \end{aligned} \tag{2.8}$$

Recall that, if $S^* = S$, $T^* = -T$, and $ST = TS$, then

$$\|S+T\|^2 \leq \|S\|^2 + \|T\|^2.$$

Thus

$$\left\|\frac{MI-(P-P^*)}{1+M}\right\|^2 = \frac{M^2 + \|P-P^*\|^2}{M^2+2M+1} \tag{2.9}$$

which is less than 1 provided $\|P-P^*\|^2 < 2M+1$, i.e., $M > \frac{1}{2}(\|P-P^*\|^2 - 1)$. In particular, if $M > \frac{1}{2}(c_0^2 - 1)$, we see that

$$Q = P\left\{\frac{1}{M+1}\left[I - \frac{MI-(P-P^*)}{1+M}\right]^{-1}\right\} \tag{2.10}$$

may be expanded in a Neumann series to give (2.5). □

We obtain, as an immediate consequence, the following:

Corollary 2.2.1. *With* $\langle P_n \rangle$ *and* $\langle Q_n \rangle$ *as in the foregoing discussion, let* B *be a real Banach function space on* $(X, d\mu)$ *such that* $\langle P_n \rangle$ *is uniformly holomorphic in a neighborhood of* 0 *in* $\mathbf{B}$. *Then* $\langle Q_n \rangle$ *is uniformly holomorphic in a neighborhood of* 0 *in* $\mathbf{B}$. □

The mappings P_n may be extended to complex-valued functions in a straightforward way: if $\beta = a + ib$ is complex valued, we define, for $n \in \mathbf{N}$,

$$P_n(\beta) = P_n(a+ib) = M_{e^{ib}} P_n(a) M_{e^{-ib}}. \tag{2.11}$$

If $P_n(\beta)$ is a bounded operator on $L(0)$ for all β in a neighborhood of 0 in $\mathbf{B}$, then the series (2.5) may be used to extend Q_n to complex-valued functions in a natural way, such that (by Proposition 2.2(b))

$$Q_n(\beta) = P_n(\beta)[I + (P_n(\beta) - P_n(-\beta))]^{-1}. \tag{2.12}$$

Now let us compute the first Gâteaux differential at 0, in the direction $\beta \in \mathbf{B}$, of P_n and Q_n. We have

$$\begin{aligned} \frac{d}{ds} P_n(s\beta)\Big|_{s=0} &= \frac{d}{ds}\{M_{e^{s\beta}} S_n(0) M_{e^{-s\beta}}\}\Big|_{s=0} \\ &= \Big\{M_\beta M_{e^{s\beta}} S_n(0) M_{e^{-s\beta}} - M_{e^{s\beta}} S_n(0) M_\beta M_{e^{-s\beta}}\Big\}\Big|_{s=0} \\ &= [M_\beta, S_n(0)]. \end{aligned} \tag{2.13}$$

Thus, by (2.12), we have

$$\begin{aligned} \frac{d}{ds} Q_n(s\beta)\Big|_{s=0} &= \frac{d}{ds} P_n(s\beta)\Big|_{s=0} - P_n(0)\frac{d}{ds}\{P_n(s\beta) - P_n(-s\beta)\}\Big|_{s=0} \\ &= [M_\beta, S_n(0)] - S_n\Big\{[M_\beta, S_n(0)] + [M_\beta, S_n(0)]\Big\} \\ &= \{I - 2S_n(0)\}[M_\beta, S_n(0)]. \end{aligned} \tag{2.14}$$

In light of these calculations we obtain

Corollary 2.2.2. *Let* $\mathbf{B}$ *be the complexification of a real Banach function space* B *on* $(X, d\mu)$, *and suppose that* $\langle P_n \rangle$ *is uniformly holomorphic in a neighborhood of* 0 *in* $\mathbf{B}$. *Then*:

(a) $\mathbf{B}$ *is the space of uniform holomorphy at* 0 *for* $\langle P_n \rangle$ *if and only if* $\beta \in \mathbf{B}$ *is a necessary and sufficient condition for*

$$\sup\{\|[M_\beta, S_n(0)]\|_{\mathscr{L}(0)} : n \in \mathbf{N}\} < \infty. \tag{2.15}$$

(b) $\mathbf{B}$ *is the space of uniform holomorphy at* 0 *for* $\langle Q_n \rangle$ *if and only if* $\beta \in \mathbf{B}$ *is a necessary and sufficient condition for*

$$\sup\{\|\{I - 2S_n(0)\}[M_\beta, S_n(0)]\|_{\mathscr{L}(0)} : n \in \mathbf{N}\} < \infty. \quad \square \tag{2.16}$$

In practice it is frequently the case that the base projections $\langle S_n(0) \rangle$ are given by integration against a kernel. In this case, the uniform holomorphy of $\langle P_n \rangle$

in a neighborhood of 0 may be reduced to the problem of obtaining a uniform weighted norm inequality for the base projections. This is a consequence of the following general result:

Proposition 2.3. *Let B be a real Banach function space on $(X, d\mu)$. Let $\langle K_n(0): n \in \mathbf{N}\rangle$ be a family of integral operators in $\mathscr{L}(0)$, and suppose that, for all $n \in \mathbf{N}$, there exists a kernel $D_n(x, y)$ such that, for $f \in L(0)$ and $x \in X$,*

$$\{K_n(0)f\}(x) = \int_X D_n(x, y)f(y)d\mu(y). \tag{2.17}$$

For each $\beta \in \mathbf{B}$, define $K_n(\beta) = M_{e^\beta}K_n(0)M_{e^{-\beta}}$. Then the following are equivalent:

(a) *$\langle K_n\rangle$ is uniformly holomorphic in a neighborhood of 0 in $\mathbf{B}$.*

(b) *There exist constants δ_0, $C_0 > 0$ such that, for every $n \in \mathbf{N}$ and for all $\beta \in B$ with $\|\beta\|_B < \delta_0$,*

$$\|K_n(\beta)\|_{\mathscr{L}(0)} \le C_0. \tag{2.18}$$

Proof. That (a) implies (b) is evident from Proposition 2.1. Now suppose (b) is true. By virtue of the fact that, for all $\alpha \in B$, the operator of multiplication by $e^{i\alpha}$ is an isometry of $L(0)$, it is clear that (b) continues to hold with 'B' replaced everywhere by '$\mathbf{B}$'. Thus, by Proposition 2.1, it suffices to show that each K_n is Gâteaux differentiable in a common neighborhood of 0 in $\mathbf{B}$. Our proof follows an idea of Coifman, Rochberg, and Weiss (see [3, §2]; see also [5, Chapter 4, Note 7.12]).

For $\beta \in \mathbf{B}$, $f \in L(0)$, and $x \in X$ we have

$$\{K_n(\beta)f\}(x) = \int_X \exp(\beta(x) - \beta(y))D_n(x, y)f(y)d\mu(y). \tag{2.19}$$

If $\alpha \in \mathbf{B}$, then the first Gâteaux differential of K_n at α in the direction β is given by

$$\begin{aligned} \left\{\frac{d}{dz}K_n(\alpha + z\beta)f\right\}(x) &= \int_X (\beta(x) - \beta(y))\exp\{\alpha(x) - \alpha(y) \\ &\qquad + z(\beta(x) - \beta(y))\}D_n(x, y)f(y)d\mu(y). \end{aligned} \tag{2.20}$$

Now let $\alpha, \beta \in \mathbf{B}$ with $\|\alpha\|_{\mathbf{B}} < \delta_0/2$ and $\|\beta\|_{\mathbf{B}} < (\delta_0/2) - \|\alpha\|_{\mathbf{B}}$. For $\theta \in [0, 2\pi]$, define the operator

$$K_{n,\theta} = K_n(\alpha + (z + e^{i\theta})\beta). \tag{2.21}$$

Now we have

$$\begin{aligned} \|\alpha + (z + e^{i\theta})\beta\|_{\mathbf{B}} &\le \|\alpha\|_{\mathbf{B}} + (1 + |z|)\|\beta\|_{\mathbf{B}} \\ &< \|\alpha\|_{\mathbf{B}} + (1 + |z|)\{(\delta_0/2) - \|\alpha\|_{\mathbf{B}}\} \\ &= (1 + |z|)(\delta_0/2) - |z|\,\|\alpha\|_{\mathbf{B}} \end{aligned} \tag{2.22}$$

which is less than δ_0 provided $|z| < 1$. Consequently, for $|z| < 1$, we have $\|K_{n,\theta}\|_{\mathscr{L}(0)} \le C_0$.

Now we claim that, for $|z| < 1$,

$$\frac{d}{dz}K_n(\alpha + z\beta) = \frac{1}{2\pi}\int_0^{2\pi} K_{n,\theta}e^{-i\theta}\,d\theta. \tag{2.23}$$

In view of (2.19)–(2.21), we see that, to establish (2.23), it suffices to show that

$$\frac{1}{2\pi}\int_0^{2\pi} \exp\{e^{i\theta}(\beta(x) - \beta(y))\}e^{-i\theta}\,d\theta = \beta(x) - \beta(y). \tag{2.24}$$

But note that, if A is a complex constant,

$$\frac{1}{2\pi}\int_0^{2\pi} \exp(Ae^{i\theta})e^{-i\theta}\,d\theta = \frac{1}{2\pi}\sum_{k=0}^{\infty}\int_0^{2\pi} \frac{(Ae^{i\theta})^k}{k!}e^{-i\theta}\,d\theta = A. \tag{2.25}$$

Letting $A = \beta(x) - \beta(y)$ in (2.25), we obtain (2.24).

From (2.23), we see that, for $|z| < 1$,

$$\left\|\frac{d}{dz}K_n(\alpha + z\beta)\right\|_{\mathscr{L}(0)} \le \frac{1}{2\pi}\int_0^{2\pi} \|K_{n,\theta}\|_{\mathscr{L}(0)}\,d\theta \le C_0. \tag{2.26}$$

From this we conclude that each K_n is Gâteaux differentiable on the open ball of radius $\delta_0/2$ in $\mathbf{B}$. □

With an additional assumption regarding the strong convergence of the operators $\langle K_n(\beta)\rangle$, we obtain the following useful result:

Proposition 2.4. *Under the hypotheses of Proposition* 2.3, *let us make the additional assumption that there is an operator* $K_\infty(0) \in \mathscr{L}(0)$ *such that, for all* β *in a neighborhood of* 0 *in* B, $K_n(\beta) - K_\infty(\beta)$ *converges to* 0 *in the strong operator topology on* $\mathscr{L}(0)$ *as* $n \to \infty$, *where* $K_\infty(\beta) = M_{e^\beta}K_\infty(0)M_{e^{-\beta}}$. *Then the following are equivalent*:

(a) *There exist constants* δ_0, $C_0 > 0$ *such that, for all* $n \in \mathbf{N}$ *and for all* $\beta \in B$ *with* $\|\beta\|_B < \delta_0$, *inequality* (2.18) *holds.*

(b) *There exist constants* δ_1, $C_1 > 0$ *such that, for all* $\beta \in B$ *with* $\|\beta\|_B < \delta_1$,

$$\|K_\infty(\beta)\|_{\mathscr{L}(0)} \le C_1. \tag{2.27}$$

(c) $\langle K_n\rangle$ *is uniformly holomorphic in a neighborhood of* 0 *in* $\mathbf{B}$.

Proof. The equivalence of (a) and (c) is simply Proposition 2.3. The equivalence of (a) and (b) is the essence of a remark made by Garnett [6, p. 109]. We give the details.

Suppose that (a) holds. Let $\varepsilon > 0$ and $\beta \in B$ with $\|\beta\|_B < \delta_0$. For every $f \in L(0)$ we can find $N > 0$ such that $n \ge N$ implies

$$\|\{K_n(\beta) - K_\infty(\beta)\}f\|_{L(0)} < \varepsilon. \tag{2.28}$$

Then, for such n, we have

$$\begin{aligned}\|K_\infty(\beta)f\|_{L(0)} &\le \|\{K_n(\beta) - K_\infty(\beta)\}f\|_{L(0)} + \|K_n(\beta)f\|_{L(0)} \\ &< \varepsilon + C_0\|f\|_{L(0)}.\end{aligned} \tag{2.29}$$

Since ε was arbitrary, we see that (b) is true with $\delta_1 = \delta_0$ and $C_1 = C_0$.

Conversely, assume that (b) is true. Let $\varepsilon > 0$ and $\beta \in B$ with $\|\beta\|_B < \delta_1$. For every $f \in L(0)$ we can find $N > 0$ such that $n \geq N$ implies (2.28). For such n, we have

$$\begin{aligned}\|K_n(\beta)f\|_{L(0)} &\leq \|\{K_n(\beta) - K_\infty(\beta)\}f\|_{L(0)} + \|K_\infty(\beta)f\|_{L(0)} \\ &< \varepsilon + C_1\|f\|_{L(0)}.\end{aligned} \tag{2.30}$$

Thus the family of operators $\{K_n(\beta): n \in \mathbf{N}, \beta \in B, \|\beta\|_B < \delta_1\}$ is "pointwise bounded" on $L(0)$. Then (a) follows from the principle of uniform boundedness. □

3. Uniform analyticity on the circle

In this section we apply our work in §2 to the case of trigonometric polynomials on the circle, $\mathbf{T}$. We parametrize $\mathbf{T}$ by the interval $[-\pi, \pi)$, and let $L(0) = L^2(\mathbf{T}) = L^2([-\pi, \pi), d\theta)$, where $d\theta$ is ordinary Lebesgue measure. Let ω be a nonnegative weight function on $\mathbf{T}$ such that $\omega^2 + \omega^{-2} \in L^1(\mathbf{T})$, and write $\beta = \log \omega$. For each integer k, define the function e_k by $e_k(\theta) = e^{ik\theta}$. We define, for $n \in \mathbf{N}$, the space

$$H_n(0) = \operatorname{span}_{\mathbf{C}}\langle e_k : |k| \leq n\rangle \tag{3.1}$$

of trigonometric polynomials of degree at most n; we define $L(\beta)$, $\mathscr{L}(\beta)$, $H_n(\beta)$, etc. as in §2. We note that the base projections $\langle S_n(0): n \in \mathbf{N}\rangle$ are simply the partial sum operators for Fourier series, defined by

$$S_n(0)f = \sum_{k=-n}^{n} \hat{f}(k)e_k \tag{3.2}$$

where, for each integer k, $\hat{f}(k)$ is the kth Fourier coefficient of f, given by

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e_{-k}(\theta)d\theta \tag{3.3}$$

and thus

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e_k \tag{3.4}$$

is the Fourier series for f. The operator $S_n(0)$ is an integral operator, given by

$$\{S_n(0)f\}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta, \psi)f(\psi)d\psi \tag{3.5}$$

where $D_n(\theta, \psi)$ is the Dirichlet kernel, given by

$$D_n(\theta, \psi) = \sum_{k=-n}^{n} e_k(\theta)e_{-k}(\psi) = \frac{\sin[(2n+1)\frac{\theta-\psi}{2}]}{\sin(\frac{\theta-\psi}{2})} \tag{3.6}$$

(see, for example, [10, p. 12]).

We shall show that, in this example, the space of functions of bounded mean oscillation on $\mathbf{T}$ is the space of uniform holomorphy at 0 for the families $\langle P_n \rangle$ and $\langle Q_n \rangle$.

We adopt the convention that I is a *subinterval* of $\mathbf{T}$ if and only if it is a subinterval of $[-\pi, \pi)$ in the usual sense, or it is the union of an interval of the form (c, π) or $[c, \pi)$ with an interval of the form $[-\pi, d)$ or $[-\pi, d]$, with $-\pi < d < c < \pi$. We let $|I|$ denote the Lebesgue measure of I. If $b \in L^1(\mathbf{T}) = L^1([-\pi, \pi), d\theta)$, we define the mean of b on I to be

$$m_I(b) = |I|^{-1} \int_I b(\theta) d\theta. \tag{3.7}$$

The function b is said to have *bounded mean oscillation* on $\mathbf{T}$ if and only if the quantity

$$\|b\|_* \equiv \sup_I |I|^{-1} \int_I |b(\theta) - m_I(b)| d\theta = \sup_I m_I(|b - m_I(b)|) \tag{3.8}$$

is finite, where the supremum is taken over all subintervals I of $\mathbf{T}$. The space $BMO(\mathbf{T})$ of real-valued functions (modulo constants) having bounded mean oscillation on $\mathbf{T}$ is a Banach space with $\|\cdot\|_*$ as its norm. For ease of notation in this section we shall refer to $BMO(\mathbf{T})$ as simply BMO; its complexification will be denoted by $\mathbf{BMO}$.

To begin, we shall show that $\langle P_n \rangle$ is uniformly holomorphic in a neighborhood of 0 in $\mathbf{BMO}$, from which it follows that the same is true of $\langle Q_n \rangle$, by Corollary 2.2.1. By Proposition 2.3, the uniform holomorphy of $\langle P_n \rangle$ in a neighborhood of 0 in $\mathbf{BMO}$ is equivalent to a uniform estimate of the form

$$\|P_n(\beta)\|_{\mathscr{L}(0)} \le C_0 \quad \text{for all } \beta \in BMO \quad \text{with } \|\beta\|_* < \delta_0, \tag{3.9}$$

where C_0, δ_0 are constants independent of n.

We can, in fact, characterize the weight functions $\omega = e^\beta$ for which $\langle \|P_n(\beta)\|_{\mathscr{L}(0)} \rangle$ is uniformly bounded in n. Recall that the weight ω is said to belong to the class A_2 if and only if

$$\sup_I m_I(\omega) m_I(\omega^{-1}) < \infty \tag{3.10}$$

where the supremum is taken over all subintervals I of $\mathbf{T}$. The quantity (3.10) is called the A_2 constant of ω. We have the following result:

Proposition 3.1. *The quantity* $\sup\langle \|P_n(\beta)\|_{\mathscr{L}(0)} : n \in \mathbf{N} \rangle$ *is finite if and only if* $\omega^2 \in A_2$.

Proof. The proof is analogous to that of [5, Corollary 3.12, Chapter 4]. The idea is to exploit the relationship between $S_n(0)$ and the orthogonal projection P_+ of $L(0)$ onto the Hardy space $\mathscr{H}_+^2 = \{f \in L(0) : \hat{f}(k) = 0 \text{ for } k < 0\}$.

Consider the operator $T_n = e_n S_n(0) e_{-n}$. A simple computation shows that, for $f \in L(0)$,

$$T_n f = \sum_{k=0}^{2n} \hat{f}(k) e_k . \tag{3.11}$$

Moreover, as $n \to \infty$, $T_n \to P_+$ in the strong operator topology on $\mathscr{L}(0)$. By a slight modification of the proof of Proposition 2.4, it follows that $\sup\langle \|M_\omega T_n M_\omega^{-1}\|_{\mathscr{L}(0)} : n \in \mathbf{N}\rangle$ is finite if and only if $M_\omega P_+ M_\omega^{-1} \in \mathscr{L}(0)$. By virtue of the relationship between P_+ and the conjugate operator (cf. [6, p. 108]), we see that $M_\omega P_+ M_\omega^{-1} \in \mathscr{L}(0)$ if and only if $\omega^2 \in A_2$ (see [7]). Now note that

$$P_n(\beta) = M_{e_{-n}} M_\omega T_n M_\omega^{-1} M_{e_n} ; \tag{3.12}$$

moreover, M_{e_k} is an isometry for each integer k. Consequently, for each integer n,

$$\|P_n(\beta)\|_{\mathscr{L}(0)} = \|M_\omega T_n M_\omega^{-1}\|_{\mathscr{L}(0)} , \tag{3.13}$$

from which the result follows. □

Corollary 3.1.1. *$\langle P_n\rangle$ and $\langle Q_n\rangle$ are uniformly holomorphic in a neighborhood of* 0 *in* **BMO**.

Proof. There exist constants δ_0, $C > 0$ such that, if $\beta \in BMO$ and $\|\beta\|_* < \delta_0$, then $\omega^2 \in A_2$, and the A_2 constant of ω^2 is less than or equal to C (see [5, Chapter 2, Corollary 3.10 and Chapter 4, Corollary 2.18]). For $\omega^2 \in A_2$, the $\mathscr{L}(0)$-norm of $M_\omega P_+ M_\omega^{-1}$ depends upon the A_2 constant of ω^2; so by the proof of Proposition 3.1, we obtain (3.9), from which the corollary follows. □

Next, we would like to show that **BMO** is actually the space of uniform holomorphy at 0 for $\langle P_n\rangle$ and $\langle Q_n\rangle$. A simple computation shows that $I - 2S_n(0)$ is an isometry of $L(0)$ for each $n \in \mathbf{N}$; so, by Corollary 2.2.2, it suffices to show that $\beta \in \mathbf{BMO}$ is a necessary and sufficient condition for boundedness of the set $\{\|[M_\beta , S_n(0)]\|_{\mathscr{L}(0)} : n \in \mathbf{N}\}$. We have the following:

Proposition 3.2. *There exist constants $C_1, C_2 > 0$ such that for all $\beta \in L^1(\mathbf{T})$,*

$$C_1\|\beta\|_* \le \sup_n \|[M_n , S_n(0)]\|_{\mathscr{L}(0)} \le C_2\|\beta\|_* . \tag{3.14}$$

Proof. The existence of C_2 with the desired property follows from the fact that $\langle P_n\rangle$ is uniformly holomorphic in a neighborhood of 0 in **BMO**. Therefore it suffices to prove that there exists a constant $\mu > 0$ such that, for all $\beta \le L^1(\mathbf{T})$,

$$\|\beta\|_* \le \mu \sup_n \|[M_\beta , S_n(0)]\|_{\mathscr{L}(0)} . \tag{3.15}$$

Let I be a subinterval of $\mathbf{T}$ and let $\theta \in [-\pi , \pi)$. Define the function $f_\theta = f_{I,\theta,n}$ by setting, for $\psi \in [-\pi , \pi)$,

$$f_\theta(\psi) = 2i \sin\left(\frac{\theta - \psi}{2}\right) \exp\left[i(2n+1)\frac{\theta-\psi}{2}\right] \chi_I(\psi) . \tag{3.16}$$

Observe that, by (3.6), when $\psi \in I$,

$$(3.17)\quad D_n(\theta, \psi) f_\theta(\psi) = 2i \sin\left[(2n+1)\frac{\theta-\psi}{2}\right] \exp\left[i(2n+1)\frac{\theta-\psi}{2}\right] \chi_I(\psi)$$
$$= 2i \cos\left[(2n+1)\frac{\theta-\psi}{2}\right] \sin\left[(2n+1)\frac{\theta-\psi}{2}\right] \chi_I(\psi)$$
$$- 2\sin^2\left[(2n+1)\frac{\theta-\psi}{2}\right] \chi_I(\psi).$$

In view of the fact that

$$(3.18)\quad \exp[i(2n+1)(\theta-\psi)]$$
$$= 1 - 2\sin^2\left[(2n+1)\frac{\theta-\psi}{2}\right]$$
$$+ 2i\cos\left[(2n+1)\frac{\theta-\psi}{2}\right] \sin\left[(2n+1)\frac{\theta-\psi}{2}\right],$$

we obtain, for $\psi \in I$,

$$(3.19)\quad D_n(\theta, \psi) f_\theta(\psi) = (\exp[i(2n+1)(\theta-\psi)] - 1)\chi_I(\psi).$$

Consequently, by (3.5) and (3.19),

$$(3.20)\quad \{[M_\beta, S_n(0)] f_\theta\}(\theta)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\exp i(2n+1)(\theta-\psi) - 1\}(\beta(\theta) - \beta(\psi))\chi_I(\psi)d\psi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[i(2n+1)(\theta-\psi)](\beta(\theta) - \beta(\psi))\chi_I(\psi)d\psi$$
$$- \frac{1}{2\pi}\int_{-\pi}^{\pi} (\beta(\theta) - \beta(\psi))\chi_I(\psi)d\psi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[i(2n+1)(\theta-\psi)](\beta(\theta) - \beta(\psi))\chi_I(\psi)d\psi$$
$$- \frac{1}{2\pi}|I|(\beta(\theta) - m_I(\beta)).$$

Now note that

$$(3.21)\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[i(2n+1)(\theta-\psi)](\beta(\theta) - \beta(\psi))\chi_I(\psi)d\psi$$
$$= \exp[i(2n+1)\theta]\left\{\beta(\theta)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(2n+1)\psi]\chi_I(\psi)d\psi \right.$$
$$\left. - \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(2n+1)\psi]\beta(\psi)\chi_I(\psi)d\psi\right\}$$
$$= \exp[i(2n+1)\theta]\{\beta(\theta)\hat{\chi}_I(2n+1) - \widehat{\beta\chi_i}(2n+1)\}.$$

Consequently, by (3.20) and (3.21),

$$(3.22)\quad \{[M_\beta, S_n(0)] f_\theta\}(\theta) = \exp[i(2n+1)\theta]\{\beta(\theta)\hat{\chi}_I(2n+1) - \widehat{\beta\chi_I}(2n+1)\}$$
$$- \frac{1}{2\pi}|I|(\beta(\theta) - m_I(\beta)).$$

Letting

$$(3.23)\quad g_n(\theta) = \chi_I(\theta)\exp[i(2n+1)\theta]\{\beta(\theta)\hat{\chi}_I(2n+1) - \widehat{\beta\chi_I}(2n+1)\},$$

$$(3.24)\qquad g(\theta) = \frac{1}{2\pi}|I|(\beta(\theta) - m_I(\beta)),$$

we see that

$$(3.25)\qquad \int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)|d\theta = \|g - g_n\|_1 .$$

Now

$$(3.26)\qquad \|g_n\|_1 \le |\hat{\chi}_I(2n+1)|\,\|\beta\|_1 + |\widehat{\beta\chi_I}(2n+1)|\,|I|;$$

the right-hand side of (3.26) tends to 0 as $n \to \infty$, by the Riemann-Lebesgue lemma, so $\lim_{n\to\infty} \|g_n\|_1 = 0$. Consequently, by Fatou's lemma,

$$(3.27)\quad \lim_{n\to\infty} \int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)|d\theta = \|g\|_1 = \frac{1}{2\pi}|I| \int_I |\beta(\theta) - m_I(\beta)|d\theta .$$

Thus (3.15) follows, once we prove an estimate of the form

$$(3.28)\qquad \overline{\lim_{n\to\infty}} \int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)|d\theta \le \frac{1}{2\pi}\mu C|I|^2$$

where

$$(3.29)\qquad C = \sup_n \|[M_\beta, S_n(0)]\|_{\mathscr{L}(0)}$$

and μ is a constant independent of β and I.

Now we have

(3.30)

$$\begin{aligned}
&\{[M_\beta, S_n(0)]f_\theta\}(\theta)\\
&\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta,\psi)(\beta(\theta) - \beta(\psi))f_\theta(\psi)d\psi\\
&\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta,\psi)(\beta(\theta) - \beta(\psi))2i\sin\left(\frac{\theta-\psi}{2}\right)\\
&\qquad\qquad \times \exp\left[i(2n+1)\frac{\theta-\psi}{2}\right]\chi_I(\psi)d\psi\\
&\quad = 2i\exp\left[i(2n+1)\frac{\theta}{2}\right]\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta,\psi)(\beta(\theta) - \beta(\psi))\sin\left(\frac{\theta-\psi}{2}\right)\\
&\qquad\qquad \times \exp\left[-i(2n+1)\frac{\psi}{2}\right]\chi_I(\psi)d\psi
\end{aligned}$$

so that

$$\begin{aligned}
(3.31)\quad &|\{[M_\beta, S_n(0)]f_\theta\}(\theta)|\\
&\quad = 2\cdot\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta,\psi)(\beta(\theta) - \beta(\psi))\sin\left(\frac{\theta-\psi}{2}\right)\right.\\
&\qquad\qquad \left.\times \exp\left[-i(2n+1)\frac{\psi}{2}\right]\chi_I(\psi)d\psi\right| .
\end{aligned}$$

Now suppose I is a subinterval of $\mathbf{T}$ which is also a subinterval of $[-\pi, \pi)$ in the ordinary sense, and let ψ_0 denote the ordinary midpoint of I. (If I is a subinterval of $\mathbf{T}$ comprising two disjoint subintervals of $[-\pi, \pi)$, we need to make minor adjustments to the argument.) Now, we write

(3.32)
$$\sin\left(\frac{\theta-\psi}{2}\right) = \sin\left(\frac{\theta-\psi_0+\psi_0-\psi}{2}\right) = \sin\left(\frac{\theta-\psi_0}{2}\right)\cos\left(\frac{\psi_0-\psi}{2}\right) + \cos\left(\frac{\theta-\psi_0}{2}\right)\sin\left(\frac{\psi_0-\psi}{2}\right),$$

so that

$$|\{[M_\beta, S_n(0)]f_\theta\}(\theta)| \le 2\left|\sin\left(\frac{\theta-\psi_0}{2}\right)\{[M_\beta, S_n(0)]h_1\}(\theta)\right| + 2\left|\cos\left(\frac{\theta-\psi_0}{2}\right)\{[M_\beta, S_n(0)]h_2\}(\theta)\right| \tag{3.33}$$

where

$$h_1(\psi) = \cos\left(\frac{\psi_0-\psi}{2}\right)\exp\left[-i(2n+1)\frac{\psi}{2}\right]\chi_I(\psi), \tag{3.34}$$

$$h_2(\psi) = \sin\left(\frac{\psi_0-\psi}{2}\right)\exp\left[-i(2n+1)\frac{\psi}{2}\right]\chi_I(\psi). \tag{3.35}$$

Now, for $\alpha \in \mathbf{R}$, we have $|\sin\alpha| \le |\alpha|\cosh\alpha$, so that, for $\psi, \theta \in I$,

$$|h_2(\psi)| \le \left|\frac{\psi_0-\psi}{2}\right|\cosh\left(\frac{\psi_0-\psi}{2}\right) \le \frac{1}{2}(\cosh\pi)|\psi_0-\psi|, \tag{3.36}$$

$$\left|\sin\left(\frac{\theta-\psi_0}{2}\right)\right| \le \left|\frac{\theta-\psi_0}{2}\right|\cosh\left(\frac{\theta-\psi_0}{2}\right) \le \frac{1}{2}(\cosh\pi)|\theta-\psi_0|. \tag{3.37}$$

Thus, by Schwarz' inequality, (3.33) and (3.36),

$$\begin{aligned}
&\frac{1}{2}\int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)|\,d\theta \\
&\le \left(\int_I \left|\sin\left(\frac{\theta-\psi_0}{2}\right)\right|^2 d\theta\right)^{1/2} \|[M_\beta, S_n(0)]h_1\|_2 \\
&\quad + \left(\int_I \left|\cos\left(\frac{\theta-\psi_0}{2}\right)\right|^2 d\theta\right)^{1/2} \|[M_\beta, S_n(0)]h_2\|_2 \\
&\le \frac{1}{2}(\cosh\pi)\left(\int_I |\theta-\psi_0|^2 d\theta\right)^{1/2} C\|h_1\|_2 + |I|^{1/2}C\|h_2\|_2 \\
&= \frac{\sqrt{3}}{12}(\cosh\pi)|I|^{3/2}C\|h_1\|_2 + |I|^{1/2}C\|h_2\|_2.
\end{aligned} \tag{3.38}$$

We have

$$\|h_1\|_2 = \left(\int_I \left|\cos\left(\frac{\psi-\psi_0}{2}\right)\right|^2 d\psi\right)^{1/2} \le |I|^{1/2}, \tag{3.39}$$

$$\|h_2\| = \left(\int_I \left| \sin\left(\frac{\psi - \psi_0}{2} \right) \right|^2 d\psi \right)^{1/2} \tag{3.40}$$
$$\leq \frac{1}{2}(\cosh \pi) \left(\int_I |\psi_0 - \psi|^2 d\psi \right)^{1/2}$$
$$= \frac{\sqrt{3}}{12}(\cosh \pi)|I|^{3/2},$$

so that

$$\int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)| d\theta \leq \frac{\sqrt{3}}{3}(\cosh \pi) C|I|^2 \tag{3.41}$$

whence

$$\varlimsup_{n\to\infty} \int_I |\{[M_\beta, S_n(0)]f_\theta\}(\theta)| d\theta \leq \frac{1}{2\pi} \left[\frac{2\pi\sqrt{3}\cosh \pi}{3} \right] C|I|^2, \tag{3.42}$$

which is (3.28), with $\mu = (2\pi\sqrt{3}\cosh \pi)/3$. This completes the proof. □

The following corollary is immediate.

Corollary 3.2.1. **BMO** *is the space of uniform holomorphy at* 0 *for* $\langle P_n \rangle$ *and* $\langle Q_n \rangle$. □

4. Uniform analyticity on $[-1, 1]$ in a neighborhood of a Jacobi weight

In this section we consider an example on the interval $[-1, 1]$. For $\gamma, \delta > -1$, we define the *Jacobi weight* with parameters γ, δ, by

$$\omega^2_{\gamma,\delta}(x) = (1-x)^\gamma (1+x)^\delta. \tag{4.1}$$

Our basic Hilbert space is

$$L_{\gamma,\delta}(0) = L^2([-1, 1], \omega^2_{\gamma,\delta}(x)dx), \tag{4.2}$$

and we shall denote the orthogonal projection onto polynomials of degree $\leq n$ in $L_{\gamma,\delta}(0)$ by $S_n^{\gamma,\delta}(0)$. We shall make a small multiplicative perturbation of $\omega^2_{\gamma,\delta}(x)dx$ by a nonnegative weight function $\omega^2 = e^{2\beta}$, where

$$\omega^2 + \omega^{-2} \in L^1([-1, 1], \omega^2_{\gamma,\delta}(x)dx), \qquad \beta \in L^1([-1, 1], dx). \tag{4.3}$$

The corresponding Hilbert space is

$$L_{\gamma,\delta}(\beta) = L^2([-1, 1], \omega^2_{\gamma,\delta}(x)\omega^2(x)dx). \tag{4.4}$$

This orthogonal projection onto polynomials of degree $\leq n$ in $L_{\gamma,\delta}(\beta)$ will be denoted by $S_n^{\gamma,\delta}(\beta)$. Setting

$$Q_n^{\gamma,\delta}(\beta) = M_\omega S_n^{\gamma,\delta}(\beta) M_\omega^{-1}, \tag{4.5}$$

$$P_n^{\gamma,\delta}(\beta) = M_\omega S_n^{\gamma,\delta}(0) M_\omega^{-1}, \tag{4.6}$$

we conjecture that the space of functions of bounded mean oscillation on $[-1, 1]$ is the space of uniform holomorphy at 0 for the families $\langle Q_n^{\gamma,\delta}\rangle$ and $\langle P_n^{\gamma,\delta}\rangle$ whenever $\gamma, \delta > -1$. In fact, we shall show that $\langle Q_n^{\gamma,\delta}\rangle$ and $\langle P_n^{\gamma,\delta}\rangle$ are uniformly holomorphic in a neighborhood of 0 in $\mathbf{BMO}([-1, 1])$ whenever $\gamma, \delta \geq -\frac{1}{2}$.

We pause to remark that the operator $S_n^{\gamma,\delta}(0)$ is the nth partial sum operator for Jacobi series with parameters γ, δ; it is an integral operator whose kernel may be expressed in terms of the Jacobi polynomials $\langle P_n^{(\gamma,\delta)}(x): n \in \mathbf{N}\rangle$, which form an orthogonal polynomial system in $L_{\gamma,\delta}(0)$ (see [10, Chapters 3 and 4]; see also [9]). Making use of Christoffel-Darboux formulas for $S_n^{\gamma,\delta}(0)$, it is possible to reduce the problem of obtaining uniform weighted norm inequalities for $S_n^{\gamma,\delta}(0)$ to that of obtaining weighted norm inequalities for the Hilbert transform, as we shall see.

We begin with some terminology. A function $b \in L^1([-1, 1], dx)$ is in $\mathbf{BMO}([-1, 1], dx)$ if and only if

$$||b||_* \equiv \sup_I |I|^{-1} \int_I |b(\theta) - m_I(b)| d\theta \tag{4.7}$$

is finite, where the supremum is taken over all subintervals I of $[-1, 1]$. $\mathbf{BMO}([-1, 1], dx)$ is a Banach space of functions (modulo constants). The space of real-valued functions in $\mathbf{BMO}([-1, 1]), dx)$ will be denoted by $BMO([-1, 1], dx)$; we will use the abbreviations $\mathbf{BMO}$ and BMO. A non-negative weight function ω belongs to the class A_2 if and only if

$$\sup_I m_I(\omega) m_I(\omega^{-1}) < \infty \tag{4.8}$$

where the supremum is taken over all subintervals I of $[-1, 1]$; the quantity (4.8) is called the A_2 constant of ω.

By analogy to the notation of §2, we let $\mathscr{L}_{\gamma,\delta}(0)$ denote the Hilbert space of bounded linear operators on $L_{\gamma,\delta}(0)$. We wish to obtain a uniform estimate of the form

$$||P_n^{\gamma,\delta}(\beta)||_{\mathscr{L}_{\gamma,\delta}(0)} \leq C \tag{4.9}$$

which is valid for all β in some neighborhood of the origin in BMO. Note that the estimate (4.9) is equivalent to an estimate of form

$$\left\| M_\omega M_{\omega_{\gamma,\delta}} S_n^{\gamma,\delta}(0) M_{\omega_{\gamma,\delta}}^{-1} M_\omega^{-1} \right\|_{\mathscr{L}(0)} \leq C, \tag{4.10}$$

where $\mathscr{L}(0) = \mathscr{L}_{0,0}(0)$, which is valid for all β in some neighborhood of the origin in BMO. In order to prove an estimate of this form, we need the following lemma.

Lemma 4.1. *Suppose $w^2 \in A_2$. Then there exist constants $\delta_1, C > 0$, depending only upon w, such that for all $\beta \in BMO$ with $||\beta||_* < \delta_1$, $e^{2\beta} w^2$ is also in A_2 with an A_2 constant less than or equal to C.*

Proof. We make use of the following characterization of A_2: a function $\varphi \in L^1([-1,1], dx)$ is the logarithm of A_2 weight if and only if the quantity

$$\sup_I |I|^{-1} \int_I \exp(|\varphi(x) - m_I(\varphi)|)dx \tag{4.11}$$

(where the supremum is taken over all subintervals I of $[-1,1]$) is finite; the quantity (4.11) is equivalent to the square root of the A_2 constant of e^φ (see [5, Chapter 4, Theorem 2.17 and Corollary 2.18]).

Now suppose $w^2 \in A_2$; let $f = \log w$. By [5, Theorem 2.7, Chapter 4] there is a constant $\varepsilon > 0$ such that $w^{2+\varepsilon} \in A_2$. Consequently, if I is a subinterval of $[-1,1]$, we have

$$\begin{aligned}
&|I|^{-1} \int_I \exp(|(2\beta + 2f)(x) - m_I(2\beta + 2f)|)dx \\
&= |I|^{-1} \int_I \exp|2\beta(x) - m_I(2\beta)| \exp|2f(x) - m_I(2f)|dx \\
&\le |I|^{-1} \left(\int_I [\exp|2\beta(x) - m_I(2\beta)|]^{(2+\varepsilon)/\varepsilon} dx \right)^{\varepsilon/(2+\varepsilon)} \\
&\quad \times \left(\int_I [\exp|2f(x) - m_I(2f)|]^{(2+\varepsilon)/2} \right)^{2/(2+\varepsilon)} \\
&= \left(|I|^{-1} \int_I \exp\left| \left(\frac{4+2\varepsilon}{\varepsilon}\right)(\beta(x) - m_I(\beta)) \right| dx \right)^{\varepsilon/(2+\varepsilon)} \\
&\quad \times \left(|I|^{-1} \int_I \exp|(2+\varepsilon)(f(x) - m_I(f))|dx \right)^{2/(2+\varepsilon)}
\end{aligned} \tag{4.12}$$

by Hölder's inequality. Now, there exist constants $\delta_0, C > 0$ such that, if $\varphi \in BMO$ and $\|\varphi\|_* < \delta_0$, then $e^{2\varphi} \in A_2$, with an A_2 constant less than or equal to C (see [5, Chapter 2, Corollary 3.10 and Chapter 4, Corollary 2.18]). Taking $\delta_1 = \varepsilon\delta_0/(2+\varepsilon)$, then $\|((2+\varepsilon)/\varepsilon)\beta\|_* < \delta_0$ when $\|\beta\|_* < \delta_1$. Consequently, if $\|\beta\|_* < \delta_1$ then (4.12) will be dominated by a constant which depends only upon w and which is independent of I. The result follows. □

We can now prove:

Proposition 4.2. *Let $\gamma, \delta \ge -\frac{1}{2}$. Then there exist constants $C, \delta_1 > 0$ such that for all $n \in \mathbf{N}$ and for all $\beta \in BMO$ with $\|\beta\|_* < \delta_1$,*

$$\|M_\omega M_{\omega_{\gamma,\delta}} S_n^{\gamma,\delta}(0) M_{\gamma,\delta}^{-1} M_\omega^{-1}\|_{\mathscr{L}(0)} \le C \tag{4.13}$$

where $\omega = e^\beta$.

Proof. We make use of Muckenhoupt's work in [9]. First of all, we note that, for $f \in L_{\gamma,\delta}(0)$,

$$S_n^{\gamma,\delta}(0)f(x) = \int_{-1}^1 K_n^{\gamma,\delta}(x,y)f(y)\omega_{\gamma,\delta}^2(y)dy, \tag{4.14}$$

where

$$K_n^{\gamma,\delta}(x,y)=\sum_{j=0}^{n}P_j^{(\gamma,\delta)}(x)P_j^{(\gamma,\delta)}(y)\|P_j^{(\gamma,\delta)}\|_{L_{\gamma,\delta}(0)}^{-2}. \tag{4.15}$$

Thus we have, for $f\in L(0)$,

(4.16)

$$M_\omega M_{\omega_{\gamma,\delta}}S_n^{\gamma,\delta}(0)M_{\omega_{\gamma,\delta}}^{-1}M_\omega^{-1}f(x)=\int_{-1}^{1}K_n^{\gamma,\delta}(x,y)\omega_{\gamma,\delta}(x)\omega_{\gamma,\delta}(y)\frac{\omega(x)}{\omega(y)}f(y)dy.$$

Using estimates from [10], Muckenhoupt writes

$$\begin{aligned}K_n^{\gamma,\delta}(x,y)&=A(n,\gamma,\delta)H_1^{\gamma,\delta}(n;x,y)\\&\quad+B(n,\gamma,\delta)[H_2^{\gamma,\delta}(n;x,y)+H_2^{\gamma,\delta}(n;y,x)]\end{aligned} \tag{4.17}$$

where $|A(n,\gamma,\delta)|$, $|B(n,\gamma,\delta)|$ are bounded above by a constant independent of n, and

$$H_1^{\gamma,\delta}(n;x,y)=(n+1)P_n^{(\gamma,\delta)}(x)P_n^{(\gamma,\delta)}(y), \tag{4.18}$$

$$H_2^{\gamma,\delta}(n;x,y)=\frac{n(1-y^2)P_n^{(\gamma,\delta)}(x)P_{n-1}^{(\gamma+1,\delta+1)}(y)}{x-y}. \tag{4.19}$$

The Jacobi polynomials satisfy

$$P_n^{(\gamma,\delta)}(x)=(-1)^nP_n^{(\gamma,\delta)}(-x),\qquad x\in[-1,1] \tag{4.20}$$

(see [10, p. 59, (4.13)]); moreover, there is a constant $K(\gamma,\delta)$ such that, for $n\in\mathbf{N}$,

$$|P_n^{(\gamma,\delta)}(x)|\le K(\gamma,\delta)n^{-1/2}(1-x+n^{-2})^{-\gamma/2-1/4},\qquad x\in[0,1] \tag{4.21}$$

(see [9, equation (2.2)] and [10, Theorem 7.32.2]). We shall use (4.20) and (4.21) to estimate

$$T_{j,n}^{\gamma,\delta}f(x)=\int_{-1}^{1}H_j^{\gamma,\delta}(n;x,y)\omega_{\gamma,\delta}(x)\omega_{\gamma,\delta}(y)\frac{\omega(x)}{\omega(y)}f(y)dy \tag{4.22}$$

for $j=1,2$, and

$$S_{2,n}^{\gamma,\delta}f(x)=\int_{-1}^{1}H_2^{\gamma,\delta}(n;y,x)\omega_{\gamma,\delta}(x)\omega_{\gamma,\delta}(y)\frac{\omega(x)}{\omega(y)}f(y)dy. \tag{4.23}$$

We shall begin by considering the operator $T_{2,n}^{\gamma,\delta}$ in some detail.

Note first that, for $n\in\mathbf{N}$ and $x,y\in[-1,1]$, we have

$$H_2^{\gamma,\delta}(n;x,y)\omega_{\gamma,\delta}(x)\omega_{\gamma,\delta}(y)=b_1^{\gamma,\delta}(n;x)b_2^{\gamma,\delta}(n;y)\frac{w(x)}{w(y)}\frac{1}{x-y} \tag{4.24}$$

where

$$w(x)=(1-x^2)^{-1/4}, \tag{4.25}$$

$$b_1^{\gamma,\delta}(n;x)=(1-x)^{\gamma/2+1/4}(1+x)^{\delta/2+1/4}n^{1/2}P_n^{(\gamma,\delta)}(x), \tag{4.26}$$

$$b_2^{\gamma,\delta}(n;y)=(1-y)^{\gamma/2+3/4}(1+y)^{\delta/2+3/4}n^{1/2}P_{n-1}^{(\gamma+1,\delta+1)}(y). \tag{4.27}$$

We shall restrict our attention to $n \geq 2$ (for $n = 0, 1$ we need only consider operators of the form $T_{1,n}^{\gamma,\delta}$; $T_{2,n}^{\gamma,\delta}$ and $S_{2,n}^{\gamma,\delta}$ need not be considered).

For $x \in [-1, 0]$, (4.20) and (4.21) imply that

$$(4.28)\quad |b_1^{\gamma,\delta}(n;x)| \leq (1-x)^{\gamma/2+1/4}(1+x)^{\delta/2+1/4}K(\delta,\gamma)(1+x+n^{-2})^{-\delta/2-1/4};$$

since $\delta \geq -\frac{1}{2}$, we have $\delta/2 + 1/4 \geq 0$ so that

$$(4.29)\qquad |b_1^{\gamma,\delta}(n;x)| \leq (1-x)^{\gamma/2+1/4}K(\gamma,\delta) \leq 2^{\gamma/2+1/4}K(\delta,\gamma).$$

For $x \in [0, 1]$, (4.21) implies that

$$(4.30)\quad |b_1^{\gamma,\delta}(n;x)| \leq (1-x)^{\gamma/2+1/4}(1+x)^{\delta/2+1/4}K(\gamma,\delta)(1-x+n^{-2})^{-\gamma/2-1/4};$$

since $\gamma \geq -\frac{1}{2}$, we have $\gamma/2 + 1/4 \geq 0$ so that

$$(4.31)\qquad |b_1^{\gamma,\delta}(n;x)| \leq (1+x)^{\delta/2+1/4}K(\gamma,\delta) \leq 2^{\delta/2+1/4}K(\gamma,\delta).$$

Similarly, for $y \in [-1, 0]$, (4.20) and (4.21) imply that

$$\begin{aligned}(4.32)\quad |b_2^{\gamma,\delta}(n;y)| &\leq (1-y)^{\gamma/2+3/4}(1+y)^{\delta/2+3/4}n^{1/2}(n-1)^{-1/2} \\ &\qquad \times K(\delta+1,\gamma+1)[1+y+(n-1)^{-2}]^{-\delta/2-3/4} \\ &\leq 2^{1/2}K(\delta+1,\gamma+1)(1-y)^{\gamma/2+3/4} \\ &\qquad \times (1+y)^{\delta/2+3/4}[1+y+(n-1)^{2}]^{-\delta/2-3/4};\end{aligned}$$

since $\delta > -1$, we have $\delta/2 + 3/4 > 1/4 \geq 0$, so that

$$(4.33)\quad |b_2^{\gamma,\delta}(n;y)| \leq 2^{1/2}K(\delta+1,\gamma+1)(1-y)^{\gamma/2+3/4} \leq 2^{\gamma/2+5/4}K(\delta+1,\gamma+1).$$

For $y \in [0, 1]$, (4.21) implies that

$$\begin{aligned}(4.34)\quad |b_2^{\gamma,\delta}(n;y)| &\leq (1-y)^{\gamma/2+3/4}(1+y)^{\delta/2+3/4}n^{1/2}(n-1)^{-1/2} \\ &\qquad \times K(\gamma+1,\delta+1)[1-y+(n-1)^{-2}]^{-\gamma/2-3/4} \\ &\leq 2^{1/2}K(\gamma+1,\delta+1)(1+y)^{\delta/2+3/4}(1-y)^{\gamma/2+3/4} \\ &\qquad \times [1-y+(n-1)^{-2}]^{-\gamma/2-3/4};\end{aligned}$$

since $\gamma > -1$, we have $\gamma/2 + 3/4 > 1/4 \geq 0$, so that

$$(4.35)\quad |b_2^{\gamma,\delta}(n;y)| \leq 2^{1/2}K(\gamma+1,\delta+1)(1+y)^{\delta/2+3/4} \leq 2^{\delta/2+5/4}K(\gamma+1,\delta+1).$$

Consequently, the functions $b_j^{\gamma,\delta}(n;\cdot)$, $j = 1, 2$, are uniformly bounded for $n \geq 2$. Letting $M_{j,n}^{\gamma,\delta}$ denote the operator of multiplication by $b_j^{\gamma,\delta}(n;\cdot)$, we have

$$(4.36)\qquad T_{2,n}^{\gamma,\delta}f = M_{1,n}^{\gamma,\delta}(M_{\omega w}HM_{\omega w}^{-1})M_{2,n}^{\gamma,\delta}\chi_{[-1,1]}f$$

where H denotes the Hilbert transform. Now it is easily seen that w^2 is an A_2 weight; by Lemma 4.1 there exist constants δ_1, $C > 0$ depending only upon

w such that if $\beta \in BMO$ with $\|\beta\|_* < \delta_1$ then $(\omega w)^2$ is also in A_2 with an A_2 constant depending only upon w. Consequently, for $\|\beta\|_* < \delta_1$,

$$\|M_{\omega w} H M_{\omega w}^{-1}\|_{\mathscr{L}(0)} \leq C \tag{4.37}$$

where C is a constant depending only upon w (see [7]). From (4.36) and (4.37) it follows that $\|T_{2,n}^{\gamma,\delta}\|_{\mathscr{L}(0)}$ is bounded by a constant independent of n and β.

The analysis of $S_{2,n}^{\gamma,\delta}$ (for $n \geq 2$) is similar. It can be seen without difficulty that

$$H_2^{\gamma,\delta}(n; y, x)\omega_{\gamma,\delta}(x)\omega_{\gamma,\delta}(y) = b_2^{\gamma,\delta}(n; x) b_1^{\gamma,\delta}(n; y) \frac{w^{-1}(x)}{w^{-1}(y)} \frac{1}{y - x} \tag{4.38}$$

so that

$$S_{2,n}^{\gamma,\delta} f = -M_{2,n}^{\gamma,\delta}(M_{\omega w^{-1}} H M_{\omega w^{-1}}^{-1}) M_{1,n}^{\gamma,\delta} \chi_{[-1,1]} f; \tag{4.39}$$

since w^{-2} is an A_2 weight, applying Lemma 4.1 as before shows that for $\|\beta\|_* < \delta_1$, $\|S_{2,n}^{\gamma,\delta}\|_{\mathscr{L}(0)}$ is bounded by a constant independent of n, β.

The analysis of $T_{1,n}^{\gamma,\delta}$ is somewhat easier. Since $\gamma, \delta \geq -\frac{1}{2}$, it is easy to see that for $x, y \in [-1, 1]$, $n \in \mathbf{N}$,

$$|H_1^{\gamma,\delta}(n; x, y)\omega_{\gamma,\delta}(x)\omega_{\gamma,\delta}(y)| \leq C(\gamma, \delta) w(x) w(y) \tag{4.40}$$

where $C(\gamma, \delta)$ is independent of n. Then, by (4.22) and (4.40),

(4.41)

$$\int_{-1}^{1} |T_{1,n}^{\gamma,\delta} f(x)|^2 dx \leq C(\gamma, \delta)^2 \int_{-1}^{1} \left| \int_{-1}^{1} w(x) w(y) \omega(x) \omega(y)^{-1} f(y) dy \right|^2 dx.$$

Letting $I = [-1, 1]$, $b = \log w$, we may write

$$\begin{aligned} w(x) w(y) \omega(x) \omega(y)^{-1} &= \exp[(b + \beta)(x) - m_I(b + \beta)] \\ &\times \exp[(b - \beta)(y) - m_I(b - \beta)] \exp[m_I(2b)]. \end{aligned} \tag{4.42}$$

Then, by (4.42) and Schwarz' inequality

$$\begin{aligned} &\int_I \left| \int_I w(x) w(y) \omega(x) \omega(y)^{-1} f(y) dy \right|^2 dx \\ &\leq \exp[m_I(4b)] \cdot \left(\int_I \exp[2(b + \beta)(x) - m_I(2(b + \beta))] dx \right) \\ &\quad \times \left(\int_I \exp[2(b - \beta)(y) - m_I(2(b - \beta))] dy \right) \|f\|_{L(0)}^2. \end{aligned} \tag{4.43}$$

When $\|\beta\|_* < \delta_1$, we have $w^2\omega^2$ and $w^2\omega^{-2} \in A_2$, so the right-hand side of (4.43) is bounded by a constant times $\|f\|_{L(0)}^2$, where the constant depends only on w. Thus $\|\beta\|_* < \delta_1$ implies that $\|T_{1,n}^{\gamma,\delta}\|_{\mathscr{L}(0)}$ is bounded by a constant independent of n, β.

By virtue of the decomposition (4.17), the proof is complete. □

The following corollaries are immediate:

Corollary 4.2.1. *$\langle P_n^{\gamma,\delta}\rangle$ and $\langle Q_n^{\gamma,\delta}\rangle$ are uniformly holomorphic in a neighborhood of* 0 *in* **BMO** *whenever $\gamma, \delta \geq -\frac{1}{2}$.* □

Corollary 4.2.2. *For $\gamma, \delta \geq -\frac{1}{2}$, there exists a constant $C(\gamma, \delta)$ such that for all $\beta \in$* **BMO**,

$$||[M_\beta, S_n^{\gamma,\delta}(0)]||_{\mathscr{L}_{\gamma,\delta}(0)} \leq C(\gamma, \delta)||\beta||_*. \quad \square \tag{4.44}$$

5. An application to the Toda flow

Let $d\mu$ be a nonnegative measure on $[-1, 1]$ which is absolutely continuous with respect to Lebesgue measure; for example, $d\mu$ may be Lebesgue measure weighted by a Jacobi weight. Following the notation of §2, for $n \in \mathbf{N}$, let $H_n(0)$ denote the set of polynomials of degree at most n, considered as a subspace of $L(0) = L^2([-1, 1], d\mu)$. Let ω be a *fixed* nonnegative real-valued function such that $\omega^2 + \omega^{-2} \in L^1([-1, 1]), d\mu)$; write $\beta = \log \omega \in L^1([-1, 1], d\mu)$. For each t in a neighborhood of 0 in $\mathbf{R}$, we consider the Gram-Schmidt procedure in the space $L(t\beta) = L^2([-1, 1], \omega^{2t} d\mu)$. Specifically, we let $\langle p_{n,t}(x) : n \in \mathbf{N}\rangle$ denote the orthogonal polynomial system on $L(t\beta)$ obtained by applying the Gram-Schmidt procedure to $\langle 1, x, x^2, \ldots\rangle$. For $t \neq 0$, it is easily seen that $\langle p_{n,t}\rangle$ arises also by applying Gram-Schmidt to $\langle p_{n,0}\rangle$; and, in fact,

$$p_{n,t} = \frac{\{S_n(t\beta) - S_{n-1}(t\beta)\}p_{n,0}}{||\{S_n(t\beta) - S_{n-1}(t\beta)\}p_{n,0}||_{L(t\beta)}}. \tag{5.1}$$

The polynomials $\langle p_{n,t}\rangle$ satisfy the following three-term recurrence (see, for example, [10, §3.2]):

$$xp_{n,t}(x) = A_{n-1}(t)p_{n-1,t}(x) + B_n(t)p_{n,t}(x) + A_n(t)p_{n+1,t}(x), \qquad n \in \mathbf{N}, \tag{5.2}$$

where we let $p_{-1,t}(x) \equiv 0 \equiv A_{-1}(t)$, and, for $n \in \mathbf{N}$,

$$A_n(t) = \int_{-1}^1 x(M_{\omega^t}p_{n,t})(x)(M_{\omega^t}p_{n+1,t})(x)d\mu(x), \tag{5.3}$$

$$B_n(t) = \int_{-1}^1 x[(M_{\omega^t}p_{n,t})(x)]^2 d\mu(x). \tag{5.4}$$

The Gram-Schmidt process can be done so that $A_n(t) > 0$ for all $n \in \mathbf{N}$. It is easy to see that, for $n \in \mathbf{N}$ and $t \in \mathbf{R}$, $|A_n(t)|$, $|B_n(t)| \leq 1$. We note for future reference that, by (5.1),

$$M_{\omega^t}p_{n,t} = \frac{\{Q_n(t\beta) - Q_{n-1}(t\beta)\}(M_{\omega^t}p_{n,0})}{||\{Q_n(t\beta) - Q_{n-1}(t\beta)\}(M_{\omega^t}p_{n,0})||_{L(0)}}. \tag{5.5}$$

Let l^2_+ denote the complex Hilbert space of square summable sequences; i.e., a sequence $\langle a_n \rangle$ is in l^2_+ if and only if

$$\|\langle a_n \rangle\|_{l^2_+} \equiv \sum_{n=0}^{\infty} |a_n|^2 < \infty; \tag{5.6}$$

the inner product on l^2_+ is given by

$$(\langle a_n \rangle , \langle b_n \rangle) = \sum_{n=0}^{\infty} a_n \overline{b}_n. \tag{5.7}$$

The mapping $L_t: L(t\beta) \to L(t\beta)$ given by $L_t f(x) = xf(x)$ induces a bounded linear transformation on l^2_+ given by the matrix

$$J(t) = \begin{pmatrix} B_0(t) & A_0(t) & 0 & 0 & \cdots & \cdots \\ A_0(t) & B_1(t) & A_1(t) & 0 & \cdots & \cdots \\ 0 & A_1(t) & B_2(t) & A_2(t) & \cdots & \cdots \\ 0 & 0 & A_2(t) & B_3(t) & A_3(t) & \cdots \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}; \tag{5.8}$$

$J(t)$ is an infinite, symmetric, tridiagonal matrix with strictly positive off-diagonal elements, i.e., a *Jacobi matrix*. The mapping $t \mapsto J(t)$ defines a flow on the space of Jacobi matrices, which is a generalized infinite-dimensional Toda flow of the type studied by Deift, Li, and Tomei in [4], especially §5. Deift et. al. have asked for a characterization of those functions ω for which the flow J is analytic in a neighborhood of 0 in $\mathbf{R}$. As an application of our earlier work, we can now give a partial answer to their question. We remark that the analyticity of the flow J is essentially equivalent to the analyticity of the Gram-Schmidt process relative to measures of the form $\omega^{2t} d\mu$ on $[-1, 1]$.

Let us be more explicit about the operator $J(t)$ on l^2_+. Suppose that $\hat{f} = \langle \hat{f}_0, \hat{f}_1, \hat{f}_2, \dots \rangle, \hat{g} = \langle \hat{g}_0, \hat{g}_1, \hat{g}_2, \dots \rangle \in l^2_+; \hat{f}$ and $\hat{g}$ give the Fourier coefficients for functions $f_t, g_t \in L(t\beta)$ defined by

$$f_t = \sum_{k=0}^{\infty} \hat{f}_k p_{k,t}, \qquad g_t = \sum_{k=0}^{\infty} \hat{g}_k p_{k,t}. \tag{5.9}$$

Then we have

(5.10)

$$\begin{aligned}(J(t)\hat{f}, \hat{g}) = {} & B_0(t)\hat{f}_0\bar{\hat{g}}_0 + A_0(t)\hat{f}_1\bar{\hat{g}}_0 \\ & + \sum_{k=0}^{\infty}\{A_k(t)\hat{f}_k\bar{\hat{g}}_{k+1} + B_{k+1}\hat{f}_{k+1}\bar{\hat{g}}_{k+1} + A_{k+1}(t)\hat{f}_{k+1}\bar{\hat{g}}_{k+1}\}.\end{aligned}$$

We state our question about the analyticity of J precisely, as follows. We would like to know: under what conditions on ω is it possible to extend J to a neighborhood U of 0 in $\mathbf{C}$, in such a way as to insure that the extension $\widetilde{J}$ is a holomorphic map from U to the space $\mathscr{L}(l_+^2)$ of bounded linear operators on l_+^2? By virtue of (5.10), we see that it suffices to obtain conditions on ω which will insure that there is a neighborhood U of 0 in $\mathbf{C}$ to which, for each $n \in \mathbf{N}$, the functions A_n and B_n can be holomorphically extended to functions of modulus ≤ 1.

The extension of A_n and B_n to complex values is easily effected by defining, for $z \in \mathbf{C}$,

$$(5.11) \qquad M_{\omega^z} p_{n,z} = \frac{\{Q_n(z\beta) - Q_{n-1}(z\beta)\}(M_{\omega^z} p_{n,0})}{\|\{Q_n(z\beta) - Q_{n-1}(z\beta)\}(M_{\omega^z} p_{n,0})\|_{L(0)}};$$

the extension is meaningful whenever the extension of $\langle Q_n \rangle$ to complex-valued functions is meaningful. Since the $L(0)$-norm of (5.11) is 1, it follows from (5.3) and (5.4) with z in place of t that $|A_n(z)|$, $|B_n(z)| \leq 1$.

In fact, it suffices to obtain conditions on ω which will guarantee the existence of a neighborhood U of 0 in $\mathbf{C}$ such that the mapping $z \mapsto M_{\omega^z} p_{n,z}$ is a holomorphic map from U to $L(0)$ for each $n \in \mathbf{N}$. In the remainder of this section, we will do this in the special case in which $d\mu$ is Lebesgue measure weighted by a Jacobi weight $\omega_{\gamma,\delta}^2$, where $\gamma, \delta \geq -\frac{1}{2}$.

We shall make use of the notation established in §4. For $n \in \mathbf{N}$ and $x \in [-1, 1]$, we let

$$(5.12) \qquad p_n^{(\gamma,\delta)}(x) = \frac{P_n^{(\gamma,\delta)}(x)}{\|P_n^{(\gamma,\delta)}\|_{L_{\gamma,\delta}}(0)}$$

denote the nth normalized Jacobi polynomial. We begin with the following result:

Lemma 5.1. *Suppose $\beta \in BMO$ and let $\omega = e^\beta$. For $n \in \mathbf{N}$, $\gamma, \delta \geq -\frac{1}{2}$, and $z \in \mathbf{C}$, let*

$$(5.13) \qquad F_n^{\gamma,\delta}(z) = M_{\omega^z} p_n^{(\gamma,\delta)}.$$

Then there exists a neighborhood U of 0 in $\mathbf{C}$, and a constant $K > 0$, such that for each $n \in \mathbf{N}$, $F_n^{\gamma,\delta}$ is an analytic function from U to $L_{\gamma,\delta}(0)$, and for all $z \in U$,

$$(5.14) \qquad \|F_n^{\gamma,\delta}(z)\|_{L_{\gamma,\delta}(0)} \leq K.$$

Proof. By virtue of (4.20), (4.21), and [10, equation (4.3.4), p. 68], we see that there is a constant $K(\gamma, \delta)$ such that, for $n \in \mathbf{N}$,

$$(5.15) \qquad |p_n^{(\gamma,\delta)}(x)| \leq \begin{cases} K(\gamma,\delta)(1 + x + n^{-2})^{-\delta/2-1/4}, & x \in [-1, 0], \\ K(\gamma,\delta)(1 - x + n)^{-2})^{-\gamma/2-1/4}, & x \in [0, 1]. \end{cases}$$

Now note that if $r, s > 1$ and $1/r + 1/s = 1$, we have

$$(5.16)\quad \|F_n^{\gamma,\delta}(z)\|_{L_{\gamma,\delta}(0)} \leq \left(\int_{-1}^{1} |\omega(x)^{2zr}|(1-x)^{\gamma}(1+x)^{\delta}\,dx\right)^{1/r} \times \left(\int_{-1}^{1} |p_n^{(\gamma,\delta)}(x)|^{2s}(1-x)^{\gamma}(1+x)^{\delta}\,dx\right)^{1/s}$$

by Hölder's inequality. Now we have, by (5.15), for $n \geq 1$,

(5.17)

$$\begin{aligned}
&\int_{0}^{1} |p_n^{(\gamma,\delta)}(x)|^{2s}(1-x)^{\gamma}(1+x)^{\delta}\,dx \\
&\quad \leq 2^{\delta}K(\gamma,\delta)^{2s}\int_{0}^{1}(1-x+n^{-2})^{-\gamma s - s/2}(1-x)^{\gamma}\,dx \\
&\quad \leq 2^{\delta+1}K(\gamma,\delta)^{2s}\left\{\int_{0}^{1-n^{-2}}(1-x)^{-\gamma s - s/2+\gamma}\,dx + \int_{1-n^{-2}}^{1} n^{2\gamma s+s}(1-x)^{\gamma}\,dx\right\} \\
&\quad = 2^{\delta+1}K(\gamma,\delta)^{2s}\left\{\int_{n^{-2}}^{1} y^{-\gamma s - s/2+\gamma}\,dy + n^{2\gamma s+s}\int_{0}^{n^{-2}} y^{\gamma}\,dy\right\}.
\end{aligned}$$

If $\gamma = -\frac{1}{2}$, the rightmost expression in (5.17) becomes

$$(5.18)\qquad 2^{\delta+1}K(\gamma,\delta)^{2s}\int_{0}^{1} y^{\gamma}\,dy = (\gamma+1)^{-1}2^{\delta+1}K(\gamma,\delta)^{2s}.$$

If $\gamma > -\frac{1}{2}$ and $1 < s < (2+2\gamma)(1+2\gamma)^{-1}$, we have $-\gamma s - s/2 + \gamma > -1$, so that the rightmost expression in (5.17) is dominated by

$$(5.19)\qquad 2^{\delta+1}K(\gamma,\delta)^{2s}\left\{\frac{2}{2\gamma+2-2\gamma s - s} + \frac{1}{\gamma+1}\right\}.$$

Thus it is not difficult to see that for $\gamma, \delta \geq \frac{1}{2}$ and for s satisfying $1 < s < \min\{(2+2\gamma)(1+2\gamma)^{-1}, (2+2\delta)(1+2\delta)^{-1}\}$, there is a constant $K_1(\gamma,\delta,s)$ such that for all $n \in \mathbf{N}$,

$$(5.20)\qquad \left(\int_{-1}^{1} |p_n^{(\gamma,\delta)}(x)|^{2s}(1-x)^{\gamma}(1+x)^{\delta}\,dx\right)^{1/s} \leq K_1(\gamma,\delta,s).$$

For any such choice of s, let r be the conjugate exponent to s. By Lemma 4.1, there exists a neighborhood U_0 of 0 in $\mathbf{C}$ such that, for all $z \in U_0$,

$$(5.21)\qquad |\omega(x)^{2zr}|(1-x)^{\gamma}(1+x)^{\delta}$$

is an A_2 weight, hence integrable on $[-1, 1]$; in fact, using the characterization of A_2 given in the proof of Lemma 4.1, it follows that for $z \in U_0$,

$$(5.22)\qquad \int_{-1}^{1} |\omega(x)^{2zr}|(1-x)^{\gamma}(1+x)^{\delta}\,dx$$

is bounded above by a constant independent of z. Thus there is a constant K such that for all $z \in U_0$, and for all $n \in \mathbf{N}$, we obtain (5.14).

Now note that

$$\frac{d}{dz}F_n^{\gamma,\delta}(z) = M_\beta M_{\omega^z} p_n^{(\gamma,\delta)}; \tag{5.23}$$

as before, if $1 < s < \min\{(2+2\gamma)(1+2\gamma)^{-1}, (2+2\delta)(1+2\delta)^{-1}\}$ and $1/r + 1/s = 1$, we obtain

$$\begin{aligned} \left\|\frac{d}{dz}F_n^{\gamma,\delta}(z)\right\|_{L_{\gamma,\delta}(0)} &\le K_1(\gamma,\delta,s)\left(\int_{-1}^1 |\beta(x)^{2zr}\omega(x)^{2zr}|(1-x)^\gamma(1+x)^\delta dx\right)^{1/r} \\ &\le K_1(\gamma,\delta,s)\left(\int_{-1}^1 \exp(4r|z\beta(x)|)\cdot(1-x)^\gamma(1+x)^\delta dx\right)^{1/r}. \end{aligned} \tag{5.24}$$

Again, there is a neighborhood V_0 of 0 in $\mathbf{C}$ such that, for all $z \in V_0$,

$$\int_{-1}^1 \exp(4r|z\beta(x)|)\cdot(1-x)^\gamma(1+x)^\delta dx \tag{5.25}$$

is bounded above by a constant independent of z. If we take $U = U_0 \cap V_0$, the lemma follows. □

Now suppose β is a fixed function in BMO, $\omega = e^\beta$, $\gamma, \delta \ge -\frac{1}{2}$. For $z \in \mathbf{C}$, define

$$M_{\omega^z} p_{n,z}^{(\gamma,\delta)} = \frac{\{Q_n^{\gamma,\delta}(z\beta) - Q_{n-1}^{\gamma,\delta}(z\beta)\}(M_{\omega^z} p_n^{(\gamma,\delta)})}{\|\{Q_n^{\gamma,\delta}(z\beta) - Q_{n-1}^{\gamma,\delta}(z\beta)\}(M_{\omega^z} p_n^{(\gamma,\delta)})\|_{L_{\gamma,\delta}(0)}}. \tag{5.26}$$

By Corollary 4.2.1 and Lemma 5.1, there is a neighborhood U_1 of 0 in $\mathbf{C}$, and a constant $K_1 > 0$, such that for $n \in \mathbf{N}$, the map

$$z \mapsto \{Q_n^{\gamma,\delta}(z\beta) - Q_{n-1}^{\gamma,\delta}(z\beta)\}(M_{\omega^z} p_n^{(\gamma,\delta)}) \tag{5.27}$$

is an analytic function from U_1 to $L_{\gamma,\delta}(0)$, and for all $z \in U_1$,

$$\|\{Q_n^{\gamma,\delta}(z\beta) - Q_{n-1}^{\gamma,\delta}(z\beta)\}(M_{\omega^z} p_n^{(\gamma,\delta)})\|_{L_{\gamma,\delta}(0)} \le K_1. \tag{5.28}$$

In particular, this implies that the family of maps (5.27) is continuous on U_1 uniformly in n. Now note that

$$\{Q_n^{\gamma,\delta}(0) - Q_{n-1}^{\gamma,\delta}(0)\}(p_n^{(\gamma,\delta)}) = p_n^{(\gamma,\delta)} \tag{5.29}$$

so that, for $z = 0$, the denominator in (5.26) is identically 1. Thus there is a neighborhood U_2 of 0 in $\mathbf{C}$, and a constant $K_2 > 0$, such that for all $z \in U_2$ and for all $n \in \mathbf{N}$,

$$\|\{Q_n^{\gamma,\delta}(z\beta) - Q_{n-1}^{\gamma,\delta}(z\beta)\}(M_{\omega^z} p_n^{(\gamma,\delta)})\|_{L_{\gamma,\delta}(0)} \ge K_2. \tag{5.30}$$

From this, then, it follows that there is a neighborhood U of 0 in $\mathbf{C}$ such that the mapping

$$z \mapsto M_{\omega^z} p_{n,z}^{(\gamma,\delta)} \tag{5.31}$$

is holomorphic from U to $L_{\gamma,\delta}(0)$ for each $n \in \mathbf{N}$. Consequently, we obtain

Proposition 5.2. *Let $\gamma, \delta \geq -\frac{1}{2}$, let $\beta \in BMO$, and let $\omega = e^{\beta}$. Let $t \mapsto J_{\gamma,\delta}(t)$ denote the Toda flow corresponding to the Gram-Schmidt process relative to the measure*

$$\omega(x)^{2t}(1-x)^{\gamma}(1+x)^{\delta}dx \tag{5.32}$$

on $[-1, 1]$. Then there is a neighborhood U of 0 in $\mathbf{C}$ to which $J_{\gamma,\delta}$ may be extended to a holomorphic map $\widetilde{J}_{\gamma,\delta}: U \to \mathscr{L}(l^2_+)$. □

6. Uniform analyticity on the circle, revisited

In §4, we showed that, for $\gamma, \delta \geq -\frac{1}{2}$, the family $\langle P_n^{\gamma,\delta}\rangle$ of conjugated partial sum operators for Jacobi series is uniformly holomorphic in a neighborhood of 0 in **BMO**. We conjecture that, in fact, **BMO** is the space of uniform holomorphy at 0 for $\langle P_n^{\gamma,\delta}\rangle$. Owing in part to the complicated form which the kernel of $P_n^{\gamma,\delta}$ takes, this conjecture will be somewhat more difficult to establish than the analogous results for the partial sums of Fourier series which we obtained in §3.

There are a number of classical results on the equiconvergence of Jacobi series with cosine series (for example, [10, Theorem 9.1.2]), which lead us to consider, as a preliminary step, the problem of determining the space of uniform holomorphy at 0 for the conjugated partial sums of cosine series. In this section, we shall show that **BMO**$([0, \pi])$—i.e., the space of even functions of bounded mean oscillation on **T**—is the space of uniform holomorphy at 0 for conjugated partial sums of cosine series.

We will use the notation established in §3, with some additions and modifications. Let $\widetilde{L}(0) = L^2([0, \pi], d\theta)$ and let ω be a nonnegative weight function on $[0, \pi]$ such that $\omega, \omega^{-1} \in \widetilde{L}(0)$; write $\beta = \log \omega$. For each $n \in \mathbf{N}$, let $\widetilde{H}_n(0)$ be the span of $\langle 1, \cos\theta, \cos 2\theta, \ldots, \cos n\theta\rangle$ in $\widetilde{L}(0)$. We define $\widetilde{L}(\beta) = L^2([0, \pi], \omega^2(\theta)d\theta)$ and let $\widetilde{H}_n(\beta)$ denote the closure of $\widetilde{H}_n(0)$ in $\widetilde{L}(\beta)$. We let $\widetilde{S}_n(\beta)$ be the self-adjoint projection of $\widetilde{L}(\beta)$ onto $\widetilde{H}_n(\beta)$, and then define

$$\widetilde{Q}_n(\beta) = M_{\omega}\widetilde{S}_n(\beta)M_{\omega}^{-1}, \tag{6.1}$$

$$\widetilde{P}_n(\beta) = M_{\omega}\widetilde{S}_n(0)M_{\omega}^{-1}. \tag{6.2}$$

A function $b \in L^1([0, \pi], d\theta)$ is an element of **BMO**$([0, \pi], d\theta)$ if and only if

$$\|b\|_* \equiv \sup_I |I|^{-1}\int_I |b(x) - m_I(b)|dx = \sup_I m_I(|b - m_I(b)|) \tag{6.3}$$

is finite, where the supremum is taken over all subintervals I of $[0, \pi]$. We shall abbreviate **BMO**$([0, \pi], d\theta)$ to $\mathbf{BMO}_e$ (the subscript 'e' stands for '*even*').

Now suppose that $g \in \widetilde{L}(0)$, and let $\tilde{g}$ be its even extension to $[-\pi, \pi)$. Then

$$S_n(0)\tilde{g}\Big|_{[0,\pi]} = \widetilde{S}_n(0)g \tag{6.4}$$

so that, for $\theta \in [0, \pi]$,

$$\{\widetilde{S}_n(0)g\}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta, \psi)\tilde{g}(\psi)d\psi = \frac{1}{2\pi}\int_0^{\pi} \widetilde{D}_n(\theta, \psi)g(\psi)d\psi \tag{6.5}$$

where, for $\psi \in [0, \pi]$,

$$\begin{aligned}\widetilde{D}_n(\theta, \psi) &= D_n(\theta, -\psi) + D_n(\theta, \psi) \\ &= \frac{\sin[(2n+1)\frac{\theta-\psi}{2}]}{\sin(\frac{\theta-\psi}{2})} + \frac{\sin[(2n+1)\frac{\theta-\psi}{2}]}{\sin(\frac{\theta+\psi}{2})}.\end{aligned} \tag{6.6}$$

Letting $\widetilde{\mathscr{L}}(0)$ denote the space of bounded linear operators on $\widetilde{L}(0)$, we obtain the following as an immediate consequence of our work in §3:

Proposition 6.1. *$\langle \widetilde{P}_n \rangle$ and $\langle \widetilde{Q}_n \rangle$ are uniformly holomorphic families of mappings from a neighborhood of* 0 *in* $\mathbf{BMO}_e$ *to* $\widetilde{\mathscr{L}}(0)$. □

To prove that $\mathbf{BMO}_e$ is actually the space of uniform holomorphy at 0 for $\langle \widetilde{P}_n \rangle$ and $\langle \widetilde{Q}_n \rangle$, it suffices by Corollary 2.2.2 to show that $\beta \in \mathbf{BMO}_e$ is a necessary and sufficient condition for boundedness of the set $\{|||[M_\beta, \widetilde{S}_n(0)]|||_{\widetilde{\mathscr{L}}(0)} : n \in \mathbf{N}\}$. To gain some intuition for this problem, we first consider a somewhat simpler problem involving "partial sums" of Fourier transforms on $\mathbf{R}$.

For $f \in L^1(\mathbf{R})$, we define the Fourier transform and its inverse according to the normalization

$$\hat{f}(\xi) = \int_{\mathbf{R}} e^{-ix\xi} f(x)dx, \qquad \check{f}(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{ix\xi} f(\xi)d\xi. \tag{6.7}$$

For each positive integer n, we define the operator $T_n(0): L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by setting

$$T_n(0)f = (\hat{f}\chi_{[-n,n]})^{\vee} \tag{6.8}$$

for $f \in L^2(\mathbf{R})$. $T_n(0)$ is a convolution operator, with kernel

$$K_n(x) = \frac{1}{\pi x}\sin nx. \tag{6.9}$$

We define the operator $\widetilde{T}_n(0): L^2([0, \infty)) \to L^2([0, \infty))$ as follows: for $g \in L^2([0, \infty))$, let $\tilde{g}$ denote its even extension to $\mathbf{R}$, and define

$$\widetilde{T}_n(0)g = T_n(0)\tilde{g}\Big|_{[0,\infty)}. \tag{6.10}$$

For $x \in [0, \infty)$, we have

$$\{\widetilde{T}_n(0)\}g(x) = \int_{\mathbf{R}} K_n(x-y)\tilde{g}(y)dy = \int_0^{\infty} \widetilde{K}_n(x, y)g(y)dy \tag{6.11}$$

where

$$\widetilde{K}_n(x, y) = K_n(x-y) + K_n(x+y) \tag{6.12}$$
$$= \frac{1}{\pi}\left\{\frac{(x+y)\sin n(x-y) + (x-y)\sin n(x+y)}{x^2-y^2}\right\}.$$

The operators $T_n(0)$ and $\widetilde{T}_n(0)$ are the continuous analogues of $S_n(0)$ and $\widetilde{S}_n(0)$, respectively. Now suppose that $\omega = e^\beta$ is a nonnegative weight function on $\mathbf{R}$ such that $\omega^2 + \omega^{-2} \in L^1_{\mathrm{loc}}(\mathbf{R})$, and define $T_n(\beta) = M_\omega T_n(0) M_\omega^{-1}$. We obtain the following continuous analogue of Proposition 3.1:

Proposition 6.2. *The quantity* $\sup\langle \|T_n(\beta)\|_{\mathscr{L}(L^2(\mathbf{R}))} : n = 1, 2, 3, \ldots\rangle$ *is finite if and only if* $\omega^2 \in A_2$.

Proof. This is the content of Corollary 3.1.2, Chapter 4 of [5]. □

Corollary 6.2.1. $\langle T_n\rangle$ *is a uniformly holomorphic family of mappings from a neighborhood of* 0 *in* $\mathbf{BMO}(\mathbf{R})$ *to* $\mathscr{L}(L^2(\mathbf{R}))$.

Proof. Completely analogous to that of Corollary 3.1.1. □

Corollary 6.2.2. $\langle \widetilde{T}_n\rangle$ *is a uniformly holomorphic family of mappings from a neighborhood of* 0 *in* $\mathbf{BMO}([0,\infty))$ *to* $\mathscr{L}(L^2([0,\infty)))$. □

It is left as a straightforward exercise for the reader to prove the continuous analogue of Proposition 3.2 (i.e., with $T_n(0)$ in place of $S_n(0)$). It is the proof of the corresponding result for $\widetilde{T}_n(0)$ that is of greatest interest to us here. We shall begin with an extremely useful lemma.

Let $1 < p < \infty$. A nonnegative weight function w on $\mathbf{R}$ is said to belong to the class A_p if and only if both w and $w^{-1/(p-1)}$ are locally integrable, and there is a constant $C > 0$ such that for all subintervals I of $\mathbf{R}$,

$$\left(|I|^{-1}\int_I w\right)\left(|I|^{-1}\int_I w^{-1/(p-1)}\right)^{p-1} \le C. \tag{6.13}$$

The smallest constant for which (6.13) holds is called the A_p constant of w. It is worth noting here that

$$A_1 = \bigcap_{1\le p<\infty} A_p, \qquad A_\infty = \bigcup_{1\le p<\infty} A_p \tag{6.14}$$

(see, for example, [5, Chapter 4, Theorem 1.14 and Corollary 2.13]). Our lemma is as follows:

Lemma 6.3. *Suppose* $w \in A_\infty$ *and* $b \in L^1_{\mathrm{loc}}(\mathbf{R})$. *Suppose, moreover, that there is a constant* $K(b)$ *such that, for all subintervals* I *of* $\mathbf{R}$,

$$w(I)^{-1}\int_I |b(x) - m_I(b)| w(x)\,dx \le K(b) \tag{6.15}$$

where

$$w(I) = \int_I w(x)\,dx. \tag{6.16}$$

Then $b \in \mathbf{BMO}(\mathbf{R})$, and $\|b\|_* \le C(w)K(b)$, where $C(w)$ is a constant depending only upon w.

Proof. Since $w \in A_\infty$, it follows from (6.14) that there is a $p \in (1, \infty)$ such that $w \in A_p$. Now let I be a subinterval of $\mathbf{R}$; we have, by Hölder's inequality,

$$(6.17)\quad \int_I |b(x) - m_I(b)|dx \le \left(\int_I |b(x) - m_I(b)|^p w(x)dx\right)^{1/p} \left(\int_I w(x)^{-1/(p-1)}dx\right)^{(p-1)/p}.$$

It is not difficult to see that there is a constant C_1 depending only upon p such that

$$(6.18)\qquad \left(\int_I |b(x) - m_I(b)|^p w(x)dx\right)^{1/p} \le C_1 w(I)^{1/p} K(b)$$

(see, for example, [5, Chapter 2, Corollary 3.10]). Moreover, if C_2 is the A_p constant of w, we have

$$(6.19)\qquad \left(\int_I w(x)^{-1/(p-1)}dx\right)^{(p-1)/p} \le C_2^{1/p}|I|w(I)^{-1/p}.$$

Combining (6.17)–(6.19), and letting $C(w) = C_1 C_2^{1/p}$, we have

$$(6.20)\qquad \int_I |b(x) - m_I(b)|dx \le |I|C(w)K(b)$$

from which the result follows. □

We make use of the lemma to prove

Proposition 6.4. *There exist constants $C_1, C_2 > 0$ such that*

$$(6.21)\qquad C_1\|\beta\|_* \le \sup_n \|[M_\beta, \widetilde{T}_n(0)]\|_{\mathrm{op}} \le C_2\|\beta\|_*$$

for all $\beta \in L^1_{\mathrm{loc}}([0, \infty))$, where $\|\cdot\|_*$ denotes the norm on $\mathbf{BMO}([0, \infty))$, and $\|\cdot\|_{\mathrm{op}}$ denotes the norm as an operator on $L^2([0, \infty))$.

Proof. The existence of C_2 with the requisite property is immediate from Corollary 6.2.2. To establish the existence of C_1, we make use of Lemma 6.3. For $x \in [0, \infty)$, let $w(x) = x^2$; w is an A_∞ weight. We shall show that there exists a constant $\mu > 0$ such that for all subintervals I of $[0, \infty)$, and for all $\beta \in L^1_{\mathrm{loc}}([0, \infty))$,

$$(6.22)\qquad w(I)^{-1}\int_I |\beta(x) - m_I(\beta)|^2 w(x)dx \le \mu C^2,$$

where

$$(6.23)\qquad C = \sup_n \|[M_\beta, \widetilde{T}_n(0)]\|_{\mathrm{op}}.$$

The result is then immediate from the $[0, \infty)$-version of Lemma 6.3.

Let I be a subinterval of $[0,\infty)$ and let $x \in [0,\infty)$. Define the function $f_x = f_{I,x,n}$ by setting,

(6.24)
$$f_x(y) = \chi_I(y)\pi(x^2 - y^2)\cos ny = \chi_I(y)\frac{\pi}{2}(x^2 - y^2)(e^{iny} + e^{-iny})$$

for $y \in [0,\infty)$. Note that, for $y \in I$,

(6.25)
$$\begin{aligned}\frac{1}{\pi}\frac{(x+y)\sin n(x-y)}{x^2-y^2}f_x(y) &= \frac{1}{4i}(x+y)[e^{in(x-y)} - e^{-in(x-y)}](e^{iny} + e^{-iny})\\ &= \frac{1}{4i}(x+y)[e^{inx} - e^{-inx} - e^{in(2y-x)} + e^{-in(2y-x)}\\ &= \frac{1}{2}(x+y)\sin nx + \frac{1}{2}(x+y)\sin n(x-2y),\end{aligned}$$

(6.26)
$$\begin{aligned}\frac{1}{\pi}\frac{(x-y)\sin n(x+y)}{x^2-y^2}f_x(y) &= \frac{1}{4i}(x-y)[e^{in(x+y)} - e^{-in(x+y)}](e^{iny} + e^{-iny})\\ &= \frac{1}{4i}(x-y)[e^{inx} - e^{-inx} - e^{-in(2y+x)} + e^{in(2y+x)}]\\ &= \frac{1}{2}(x-y)\sin nx + \frac{1}{2}(x-y)\sin n(x+2y),\end{aligned}$$

so that, by (6.12),

$$\begin{aligned}K_n(x,y)f_x(y) &= \{x\sin nx + \tfrac{1}{2}(x+y)\sin n(x-2y)\\ &\qquad + \tfrac{1}{2}(x-y)\sin n(x+2y)\}\chi_I(y)\\ &= \{x\sin nx + x\sin nx\cos 2ny - y\sin 2ny\cos nx\}\chi_I(y).\end{aligned}$$

Combining (6.11) and (6.27), we obtain

(6.28)
$$\begin{aligned}\{[M_\beta, \widetilde{T}_n(0)]f_x\}(x) &= x\sin nx|I|(\beta(x) - m_I(\beta))\\ &\quad + x\sin nx\int_0^\infty \cos 2ny(\beta(x) - \beta(y))\chi_I(y)dy\\ &\quad - \cos nx\int_0^\infty y\sin 2ny(\beta(x) - \beta(y))\chi_I(y)dy.\end{aligned}$$

As in the proof of Proposition 3.2, we apply the Riemann-Lebesgue lemma and Fatou's lemma to see that

(6.29)
$$\begin{aligned}\overline{\lim_{n\to\infty}}\int_I |\{[M_\beta, \widetilde{T}_n(0)]f_x\}(x)|^2 dx &= \overline{\lim_{n\to\infty}}\int_I x^2\sin^2 nx|I|^2|\beta(x) - m_I(\beta)|^2 dx\\ &= \frac{1}{2}|I|^2\int_I |\beta(x) - m_I(\beta)|^2 w(x)dx\end{aligned}$$

where we have used the fact that $\sin^2 nx = \frac{1}{2} - \frac{1}{2}\cos 2nx$.

Now suppose x_0 is the midpoint of I, and suppose further that $x_0 > 2|I|$. By (6.24), we may write

(6.30)
$$f_x(y) = (x^2 - x_0^2)\frac{\pi}{2}\chi_I(y)\cos ny + (x_0^2 - y^2)\frac{\pi}{2}\chi_I(y)\cos ny$$

so that

$$\{[M_\beta, \tilde{T}_n(0)]f_x\}(x) = (x^2 - x_0^2)\frac{\pi}{2}\{[M_\beta, \tilde{T}_n(0)]h_1\}(x) + \frac{\pi}{2}\{[M_\beta, \tilde{T}_n(0)]h_2\}(x), \tag{6.31}$$

where

$$h_1(y) = \chi_I(y)\cos ny, \qquad h_2(y) = (x_0^2 - y^2)\chi_I(y)\cos ny. \tag{6.32}$$

If $x_0 > 2|I|$, then, for $y \in I$,

$$\frac{3x_0}{4} \le x_0 - \frac{|I|}{2} \le y \le x_0 + \frac{|I|}{2} \le \frac{5x_0}{4}, \tag{6.33}$$

so that, in particular,

$$(y^2 - x_0^2)^2 = (y - x_0)^2(y + x_0)^2 \le (|I|^2/4)(81x_0^2/16) \le 2x_0^2|I|^2. \tag{6.34}$$

Consequently,

$$\|h_1\|_2^2 \le |I|, \tag{6.35}$$

$$\|h_2\|_2^2 \le \int_I 2x_0^2|I|^2 dx = 2x_0^2|I|^3, \tag{6.36}$$

$$w(I) = \int_I y^2 dy \ge \frac{9}{16}x_0^2|I|. \tag{6.37}$$

Thus, by (6.23), (6.31), and (6.34)–(6.37), we have

$$\int_I |\{[M_\beta, \tilde{T}_n(0)]f_x\}(x)|^2 dx \le \pi^2 x_0^2 |I|^3 C^2 \le \frac{16}{9}\pi^2 C^2 w(I)|I|^2. \tag{6.38}$$

Thus, by (6.29)

$$w(I)^{-1}\int_I |\beta(x) - m_I(\beta)|^2 w(x)dx \le \frac{32}{9}\pi^2 C^2. \tag{6.39}$$

If, on the other hand, $\frac{1}{2}|I| \le x_0 \le 2|I|$, we write

$$f_x(y) = x^2\frac{\pi}{2}\chi_I(y)\cos ny - y^2\frac{\pi}{2}\chi_I(y)\cos ny \tag{6.40}$$

so that

$$\{[M_\beta, \tilde{T}_n(0)]f_x\}(x) = x^2\frac{\pi}{2}\{[M_\beta, \tilde{T}_n(0)]h_1\}(x) - \frac{\pi}{2}\{[M_\beta, \tilde{T}_n(0)]wh_1\}(x). \tag{6.41}$$

Now we have

$$\begin{aligned} w(I) = \int_I y^2 dy &= \tfrac{1}{3}\left\{(x_0 + \tfrac{1}{2}|I|)^3 - (x_0 - \tfrac{1}{2}|I|)^3\right\} \\ &= \tfrac{1}{3}\left\{3x_0^2|I| + \tfrac{1}{4}|I|^3\right\} \\ &\ge \tfrac{1}{12}|I|^3, \end{aligned} \tag{6.42}$$

$$\|h_1\|_2^2 \le |I|, \tag{6.43}$$

$$\|wh_1\|_2^2 \le \int_I y^4 dy \le \tfrac{1}{5}|I|^5. \tag{6.44}$$

Noting that, for $x \in I$, $x \le \frac{5}{2}|I|$, we have

$$\int_I |\{[M_\beta, \widetilde{T}_n(0)]f_x\}(x)|^2 dx \le 10\pi^2 C^2 |I|^5 \le 120\pi^2 C^2 w(I)|I|^2, \tag{6.45}$$

by (6.23) and (6.41)–(6.44). Hence, by (6.29),

$$w(I)^{-1} \int_I |\beta(x) - m_I(\beta)|^2 w(x) dx \le 240\pi^2 C^2. \tag{6.46}$$

Thus, combining (6.39) and (6.46), we obtain (6.22) with $\mu = 240\pi^2$. This completes the proof. □

Corollary 6.4.1. **BMO**$([0, \infty))$ *is the space of uniform holomorphy at* 0 *for* $\langle \widetilde{T}_n \rangle$. □

The proof of the periodic analogue of Proposition 6.4 is only slightly more complicated, but the basic idea is the same:

Proposition 6.5. *Suppose* $\beta \in L^2([0, \pi])$. *Then*

$$C = \sup_n \|[M_\beta, \widetilde{S}_n(0)]\|_{\widetilde{\mathscr{L}}(0)} < \infty \tag{6.47}$$

if and only if $\beta \in \mathbf{BMO}_e$.

Proof. The sufficiency of $\beta \in \mathbf{BMO}_e$ is an immediate consequence of Proposition 6.1. We shall prove that (6.47) implies $\beta \in \mathbf{BMO}_e$ by using the $[0, \pi]$-version of Lemma 6.3.

For $\theta \in [0, \pi]$, let $w(\theta) = \sin^2 \theta/2$; w is an A_∞ weight on $[0, \pi]$. We shall show that there exists a constant $\mu > 0$ such that whenever I is a subinterval of $[0, \pi]$ with $|I| < \pi/400$, and whenever $\beta \in L^1([0, \pi])$, there is a constant $c_I(\beta)$ such that

$$w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \le \mu C^2. \tag{6.48}$$

The result then follows from the $[0, \pi]$-version of Lemma 6.3 together with [5, Chapter 1, Lemmas 9.4 and 9.5].

Let I be a subinterval of $[0, \pi]$ with $|I| < \pi/400$, and let $\theta \in I$. Define the function $f_\theta = f_{I,\theta,n}$ by setting, for $\psi \in [0, \pi]$,

$$\begin{aligned} f_\theta(\psi) &= 2\chi_I(\psi) \cos\left[(2n+1)\frac{\psi}{2}\right] \sin\left(\frac{\theta - \psi}{2}\right) \sin\left(\frac{\theta + \psi}{2}\right) \\ &= \chi_I(\psi)\left\{\exp\left[i(2n+1)\frac{\psi}{2}\right] + \exp\left[-i(2n+1)\frac{\psi}{2}\right]\right\} \\ &\quad \times \sin\left(\frac{\theta - \psi}{2}\right) \sin\left(\frac{\theta + \psi}{2}\right). \end{aligned} \tag{6.49}$$

A straightforward calculation using (6.6) shows that

$$\widetilde{D}_n(\theta, \psi) f_\theta(\psi) = 2 \sin\left[(2n+1)\frac{\theta}{2}\right] \sin\frac{\theta}{2} \cos\frac{\psi}{2} + \Phi(n, \theta, \psi) \tag{6.50}$$

where

$$\begin{aligned} \Phi(n, \theta, \psi) = {} & \sin\left[(2n+1)\left(\frac{\theta - 2\psi}{2}\right)\right] \sin\left(\frac{\theta+\psi}{2}\right) \\ & + \sin\left[(2n+1)\left(\frac{\theta + 2\psi}{2}\right)\right] \sin\left(\frac{\theta-\psi}{2}\right). \end{aligned} \tag{6.51}$$

For $\psi \in [0, \pi]$, let $\rho(\psi) = \cos \psi/2$, and set

$$\rho(I) = \int_I \rho(\psi) d\psi. \tag{6.52}$$

Then, by (6.5) and (6.50),

$$\begin{aligned} \{[M_\beta, \widetilde{S}_n(0)]f_\theta\}(\theta) &= \frac{1}{2\pi} \int_0^\pi \widetilde{D}_n(\theta, \psi) f_\theta(\psi)(\beta(\theta) - \beta(\psi)) d\psi \\ &= \frac{1}{\pi} \sin\left[(2n+1)\frac{\theta}{2}\right] \sin\frac{\theta}{2} \rho(I)[\beta(\theta) - c_I(\beta)] \\ &\quad + \frac{1}{2\pi} \int_I \Phi(n, \theta, \psi)(\beta(\theta) - \beta(\psi)) d\psi, \end{aligned} \tag{6.53}$$

where

$$c_I(\beta) = \rho(I)^{-1} \int_I \beta(\psi) \rho(\psi) d\psi \tag{6.54}$$

and, by the Riemann-Lebesgue lemma,

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_I \Phi(n, \theta, \psi)(\beta(\theta) - \beta(\psi)) d\psi = 0. \tag{6.55}$$

Applying the Riemann-Lebesgue lemma and Fatou's lemma in by now familiar fashion, we obtain

$$\begin{aligned} & \overline{\lim_{n\to\infty}} \int_I |\{[M_\beta, \widetilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \\ &= \overline{\lim_{n\to\infty}} \frac{1}{\pi^2} \rho(I)^2 \int_I \sin^2\left[(2n+1)\frac{\theta}{2}\right] |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \\ &= \frac{1}{2\pi^2} \rho(I)^2 \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta) d\theta \end{aligned} \tag{6.56}$$

where we have used the fact that $\sin^2[(2n+1)\frac{\theta}{2}] = \frac{1}{2} - \frac{1}{2}\cos[(2n+1)\theta]$.

Now suppose ψ_0 is the midpoint of I. We shall consider five cases:

(i) $\pi/200 > 2|I| \geq \psi_0$;

(ii) $\pi/200 > \psi_0 > 2|I|$;

(iii) $\pi/200 \leq \psi_0 \leq 199\pi/200$;

(iv) $\pi/200 > 2|I| \geq \pi - \psi_0$, i.e., $199\pi/200 < \pi - 2|I| \leq \psi_0$;

(v) $\pi/200 > \pi - \psi_0 > 2|I|$, i.e., $199\pi/200 < \psi_0 < \pi - 2|I|$.

We begin with:

Case (i): $\pi/200 > 2|I| \geq \psi_0$. The addition formula for sines show that

$$\sin\left(\frac{\theta+\psi}{2}\right)\sin\left(\frac{\theta-\psi}{2}\right) = \sin^2\frac{\theta}{2}\cos^2\frac{\psi}{2} - \sin^2\frac{\psi}{2}\cos^2\frac{\theta}{2} = w(\theta)\rho^2(\psi) - w(\psi)\rho^2(\theta). \tag{6.57}$$

If we let

$$\lambda_n(\psi) = 2\cos\left[(2n+1)\frac{\psi}{2}\right]\chi_I(\psi) \tag{6.58}$$

then, by (6.49),

$$f_\theta(\psi) = w(\theta)\rho^2(\psi)\lambda_n(\psi) - \rho^2(\theta)w(\psi)\lambda_n(\psi) \tag{6.59}$$

so that

(6.60)

$$\{[M_\beta, \widetilde{S}_n(0)]f_\theta\}(\theta) = w(\theta)\{[M_\beta, \widetilde{S}_n(0)]\rho^2\lambda_n\}(\theta) - \rho^2(\theta)\{[M_\beta, \widetilde{S}_n(0)]w\lambda_n\}(\theta).$$

It is easily seen that there exist constants $C_1, C_2, C_3 > 0$ such that, for $\psi \in I$,

$$C_I|\psi|^2 \leq w(\psi) \leq C_2|\psi|^2 \quad \text{and} \quad C_3 \leq \rho(\psi) \leq 1. \tag{6.61}$$

Thus we have

$$|I| \geq \rho(I) \geq C_3|I|, \tag{6.62}$$

$$||\rho^2\lambda_n||_2^2 \leq \int_I \rho(\psi)^4 d\psi \leq |I|, \tag{6.63}$$

$$||w\lambda_n||_2^2 \leq C_2^2 \int_I \psi^4 d\psi = C_2^2 \tfrac{1}{5}(\tfrac{5}{2}|I|)^5 \leq 20C_2^2|I|^5, \tag{6.64}$$

$$w(I) \geq C_1 \int_I \psi^2 d\psi \geq \tfrac{1}{12}C_1|I|^3, \tag{6.65}$$

where the estimation (6.65) is exactly like (6.42). Note also that, for $\psi \in I$,

$$w(\psi) \leq C_2|\psi|^2 \leq \tfrac{25}{4}C_2|I|^2. \tag{6.66}$$

Then we have

(6.67)

$$\begin{aligned}
&\int_I \{[M_\beta, \widetilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \\
&\quad\leq \int_I |w(\theta)|^2|\{[M_\beta, \widetilde{S}_n(0)]\rho^2\lambda_n\}(\theta)|^2 d\theta + \int_I |\{[M_\beta, \widetilde{S}_n(0)]w\lambda_n\}(\theta)|^2 d\theta \\
&\quad\leq \left[\tfrac{25}{4}C_2|I|^2\right]^2 \cdot C^2 \cdot ||\rho^2\lambda_n||_2^2 + C^2 \cdot ||w\lambda_n||_2^2 \\
&\quad\leq K_1 w(I)\rho(I)^2 C^2
\end{aligned}$$

where K_1 is independent of β, I, and n. Combining (6.56) and (6.67), we obtain

$$w(I)^{-1}\int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta)d\theta \leq 2\pi^2 K_1 C^2. \tag{6.68}$$

Case (ii): $\pi/200 > \psi_0 > 2|I|$. In this case we write

(6.69)
$$\sin\left(\frac{\theta-\psi}{2}\right) = \sin\left(\frac{\theta-\psi_0+\psi_0-\psi}{2}\right) = \sin\left(\frac{\theta-\psi_0}{2}\right)\cos\left(\frac{\psi_0-\psi}{2}\right) + \sin\left(\frac{\psi_0-\psi}{2}\right)\cos\left(\frac{\theta-\psi_0}{2}\right),$$

(6.70)
$$\sin\left(\frac{\theta+\psi}{2}\right) = \sin\frac{\theta}{2}\cos\frac{\psi}{2} + \cos\frac{\theta}{2}\sin\frac{\psi}{2} = w^{1/2}(\theta)\rho(\psi) + \rho(\theta)w^{1/2}(\psi).$$

If we let $w_0^{1/2}(\theta) = \sin(\frac{\theta-\psi_0}{2})$, $\rho_0(\theta) = \cos(\frac{\theta-\psi_0}{2})$, then we have

(6.71)
$$\begin{aligned}\sin\left(\frac{\theta+\psi}{2}\right)&\sin\left(\frac{\theta-\psi}{2}\right)\\ &= [w^{1/2}(\theta)\rho(\psi) + \rho(\theta)w^{1/2}(\psi)][w_0^{1/2}(\theta)\rho_0(\psi) - w_0^{1/2}(\psi)\rho_0(\theta)]\\ &= (ww_0)^{1/2}(\theta)(\rho\rho_0)(\psi) - (w^{1/2}\rho_0)(\theta)(\rho w_0^{1/2})(\psi)\\ &\quad + (w_0^{1/2}\rho)(\theta)(\rho_0 w^{1/2})(\psi) - (\rho\rho_0)(\theta)(ww_0)^{1/2}(\psi)\end{aligned}$$

so that, by (6.49), (6.58), and (6.71), we have

(6.72)
$$\begin{aligned}\{[M_\beta, \widetilde{S}_n(0)]f_\theta\}(\theta) &= (ww_0)^{1/2}(\theta)\{[M_\beta, \widetilde{S}_n(0)]\lambda_n\rho\rho_0\}(\theta)\\ &\quad - (w^{1/2}\rho_0)(\theta)\{[M_\beta, \widetilde{S}_n(0)]\lambda_n\rho w_0^{1/2}\}(\theta)\\ &\quad + (w_0^{1/2}\rho)(\theta)\{[M_\beta, \widetilde{S}_n(0)]\lambda_n\rho_0 w^{1/2}\}(\theta)\\ &\quad - (\rho\rho_0)(\theta)\{[M_\beta, \widetilde{S}_n(0)]\lambda_n w^{1/2}w_0^{1/2}\}(\theta).\end{aligned}$$

Since ψ_0 and $|I|$ are small, (6.61) continues to hold for $\psi \in I$, and hence (6.62) holds as well. Moreover, (6.61) holds for w_0, ρ_0 in place of w, ρ. As in (6.33), we obtain

(6.73)
$$\frac{3\psi_0}{4} \le \psi \le \frac{5\psi_0}{4}, \qquad \psi \in I.$$

Thus we have:

(6.74)
$$w(I) \ge \frac{9}{16}C_1\psi_0^2|I|.$$

For $\psi \in I$, we have

(6.75)
$$|(ww_0)^{1/2}(\psi)| \le \tfrac{5}{8}C_2\psi_0|I|,$$

(6.76)
$$|(w^{1/2}\rho_0)(\psi)| \le \tfrac{5}{4}C_2^{1/2}\psi_0,$$

(6.77)
$$|(w_0^{1/2}\rho)(\psi)| \le \tfrac{1}{2}C_2^{1/2}|I|,$$

(6.78)
$$|(\rho\rho_0)(\psi)| \le 1,$$

and so we have

(6.79) $$\|\lambda_n(ww_0)^{1/2}\|_2^2 \le \tfrac{25}{64}C_2^2\psi_0^2|I|^3,$$

(6.80) $$\|\lambda_n(w^{1/2}\rho_0)\|_2^2 \le \tfrac{25}{16}C_2\psi_0^2|I|,$$

(6.81) $$\|\lambda_n(w_0^{1/2}\rho))\|_2^2 \le \tfrac{1}{4}C_2|I|^3,$$

(6.82) $$\|\lambda_n(\rho\rho_0))\|_2^2 \le |I|.$$

Thus, by (6.62), (6.72), and (6.74)–(6.82) we have

(6.83) $$\int_I |\{[M_\beta, \widetilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \le KC^2|I|^3\psi_0^2 \le K_2C^2w(I)\rho(I)^2$$

where K, K_2 are independent of β, I, n. Combining (6.56) and (6.83), we obtain

(6.84) $$w(I)^{-1}\int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta)d\theta \le 2\pi^2K_2C^2.$$

Case (iii): $\pi/200 \le \psi_0 \le 199\pi/200$. Exactly as in Case (ii), we obtain the expression (6.72). For this case, we observe that the functions w, ρ_0, and ρ behave essentially as constants. Moreover, there is a constant C_4 such that for $\psi \in I$,

(6.85) $$|w_0(\psi)| \le C_4|I|^2.$$

Thus we have

(6.86) $$w(I) \sim |I|,$$

(6.87) $$\rho(I)^2 \sim |I|^2,$$

(6.88) $$\|\lambda_n(ww_0)^{1/2}\|_2^2 \lesssim |I|^2,$$

(6.89) $$\|\lambda_n(w_0^{1/2}\rho_0)\|_2^2 \lesssim |I|,$$

(6.90) $$\|\lambda_n(w_0^{1/2}\rho)\|_2^2 \lesssim |I|^2,$$

(6.91) $$\|\lambda_n(\rho\rho_0)\|_2^2 \lesssim |I|$$

so that, combining (6.72) and (6.85)–(6.91) we have

(6.92) $$\int_I |\{[M_\beta, \widetilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \le K_3C^2w(I)\rho(I)^2$$

where K_3 is independent of β, I, n. Combining (6.56) and (6.92), we have

(6.93) $$w(I)^{-1}\int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta)d\theta \le 2\pi^2K_3C^2.$$

Case (iv): $\pi/200 > 2|I| \ge \pi - \psi_0$. This case is in certain respects analogous to Case (i). Using the fact that $\sin x = \sin(\pi - x)$, we write, as in (6.57),

(6.94)
$$\sin\left(\frac{\theta+\psi}{2}\right)\sin\left(\frac{\theta-\psi}{2}\right) = \sin\left[\frac{(\pi-\theta)+(\pi-\psi)}{2}\right]\sin\left[\frac{(\pi-\theta)-(\pi-\psi)}{2}\right]$$
$$= w_\pi(\theta)\rho_\pi^2(\psi) - w_\pi(\psi)\rho_\pi^2(\theta)$$

where $w_\pi(\psi) = w(\pi - \psi)$, $\rho_\pi(\psi) = \rho(\pi - \psi)$. Thus we obtain (6.60) with w_π in place of w and ρ_π in place of ρ. The estimate involving w_π and ρ_π are essentially the same as those involving w and ρ in Case (i). Moreover, it is easy to see that $w(I) \sim |I|$ while $\rho(I)^2 \gtrsim |I|^4$, so that $w(I)\rho(I)^2 \gtrsim |I|^5$. Thus, by essentially the same argument as in Case (i), we have

$$w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta)d\theta \le 2\pi^2 K_4 C^2 \tag{6.95}$$

where K_4 is independent of β, I, n.

Case (v): $\pi/200 > \pi - \psi_0 > 2|I|$. This case is analogous to Case (ii). We write

$$\sin\left(\frac{\theta - \psi}{2}\right) = w_0^{1/2}(\theta)\rho_0(\psi) - w_0^{1/2}(\psi)\rho_0(\theta) \tag{6.96}$$

(6.97)

$$\sin\left(\frac{\theta + \psi}{2}\right) = \sin\left[\frac{(\pi - \theta)}{2} + \frac{(\pi - \psi)}{2}\right] = w_\pi^{1/2}(\theta)\rho_\pi(\psi) + \rho_\pi(\theta) w_\pi^{1/2}(\psi)$$

and, as in Case (ii), we obtain (6.72) with w_π in place of w and ρ_π in place of ρ.

Geometrically, I is a "mirror image" across $\psi = \pi/2$ of an interval of the type considered in Case (ii). Consequently, we obtain estimates of the form

$$\rho(I) \gtrsim (\pi - \psi_0)|I|, \qquad w(I) \sim |I|, \tag{6.98}$$

$$|(w_\pi w_0)^{1/2}(\psi)| \lesssim |I| \quad \text{for } \psi \in I, \tag{6.99}$$

$$|(w_\pi^{1/2}\rho_0)(\psi)| \le 1 \quad \text{for } \psi \in I, \tag{6.100}$$

$$|(w_0^{1/2}\rho_\pi)(\psi)| \lesssim (\pi - \psi_0)|I| \quad \text{for } \psi \in I, \tag{6.101}$$

$$|(\rho_\pi \rho_0)(\psi)| \lesssim (\pi - \psi_0) \quad \text{for } \psi \in I \tag{6.102}$$

so that

$$\|\lambda_n (w_\pi w_0)^{1/2}\|_2^2 \lesssim |I|^3, \tag{6.103}$$

$$\|\lambda_n (w_\pi^{1/2}\rho_0)\|_2^2 \le |I|, \tag{6.104}$$

$$\|\lambda_n (w_0^{1/2}\rho_\pi)\|_2^2 \lesssim (\pi - \psi_0)^2 |I|^3, \tag{6.105}$$

$$\|\lambda_n (\rho_\pi \rho_0)\|_2^2 \lesssim (\pi - \psi_0)^2 |I|. \tag{6.106}$$

Thus we have

$$\int_I |\{[M_\beta, \tilde{S}_n(0)]f_\theta\}(\theta)|^2 d\theta \le K' C^2 |I|^3 (\pi - \psi_0)^2 \le K_5 C^2 w(I)\rho(I)^2 \tag{6.107}$$

where K', K_5 are independent of β, I, n. Combining (6.56) and (6.107), we obtain

$$w(I)^{-1} \int_I |\beta(\theta) - c_I(\beta)|^2 w(\theta)d\theta \le 2\pi^2 K_5 C^2. \tag{6.108}$$

We have now considered all possible cases. If we take $\delta = 2\pi^2 \max_{1\leq j\leq 5} K_j$, we obtain (6.48), and the proof is complete. □

Corollary 6.5.1. $\mathbf{BMO}_e$ *is the space of uniform holomorphy at* 0 *for* $\langle\widetilde{P}_n\rangle$ *and* $\langle\widetilde{Q}_n\rangle$. □

We also obtain the following result for partial sum operators on $[-1, 1]$:

Corollary 6.5.2. *In the notation of* § 4, **BMO** *is the space of uniform holomorphy at* 0 *for* $\langle P_n^{-1/2,-1/2}\rangle$ *and* $\langle Q_n^{-1/2,-1/2}\rangle$.

Proof. By Corollary 4.2.1, it suffices to show that

$$C = \sup_n \|[M_\beta, S_n^{-1/2,-1/2}(0)\|_{\mathscr{L}_{-1/2,-1/2}(0)} < \infty, \tag{6.109}$$

for $\beta \in L^1([-1, 1], (1-x^2)^{-1/2}dx)$, only if $\beta \in \mathbf{BMO}$.

For $f \in L_{-1/2,-1/2}(0)$ and $\theta \in [0, \pi]$, define $Uf(\theta) = f\circ\cos(\theta)$. It is easy to see that U is an isometry from $L_{-1/2,-1/2}(0)$ to $\widetilde{L}(0)$. By virtue of this isometry, we see that the family $\langle t_n : n \in \mathbf{N}\rangle$ of Chebyshev polynomials, defined by

$$t_n(x) = \cos n\theta, \qquad x = \cos\theta \tag{6.110}$$

is an orthogonal polynomial system on $[-1, 1]$ relative to $(1-x^2)^{-1/2}$ (cf. [10], §§1.12 and 2.4). In particular, we see that

$$S_n^{-1/2,-1/2}(0) = U^{-1}\widetilde{S}_n(0)U \tag{6.111}$$

so that

$$[M_\beta, S_n^{-1/2,-1/2}(0)] = U^{-1}[M_{\beta\circ\cos}, \widetilde{S}_n(0)]U. \tag{6.112}$$

Since U, U^{-1} are isometries, (6.109) and (6.112) imply that $\beta\circ\cos \in \mathbf{BMO}_e$.

Now suppose $I = [a, b]$ is any subinterval of $[-1, 1]$, and let $\omega(x) = (1-x^2)^{-1/2}$; clearly ω is an A_∞ weight on $[-1, 1]$. Let $J = [\arccos b, \arccos a]$; then we have

$$\begin{aligned}\omega(I)^{-1}&\int_I |\beta(x) - m_J(\beta\circ\cos)|\omega(x)dx \\ &= |J|^{-1}\int_J |\beta\circ\cos(\theta) - m_J(\beta\circ\cos)|d\theta \leq \|\beta\circ\cos\|_*.\end{aligned} \tag{6.113}$$

Thus $\beta \in \mathbf{BMO}$ by the $[-1, 1]$-version of Lemma 6.3. □

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